Journal of Pure and Applied Algebra 213 (2009) 947–968

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Polygraphic resolutions and homology of monoids[☆]

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ARTICLE INFO

Article history:
Received 18 July 2006
Received in revised form 8 July 2008
Available online 11 November 2008
Communicated by I. Moerdijk

MSC:
18D05
18G10

ABSTRACT

We prove that for any monoid M, the homology defined by the second author by means of polygraphic resolutions coincides with the homology classically defined by means of resolutions by free $\mathbb{Z}M$-modules.

1. Introduction

Since the work of Squier and others [1,21,9], we know that monoids presented by a finite, terminating and confluent rewriting system satisfy a homological finiteness condition. This has two consequences:

- the possibility to prove negative results, e.g. examples of monoids having a decidable word problem, but no presentation satisfying the above conditions;
- on the positive side, the construction of explicit resolutions from such presentations. See for example [3] for a recent application of similar methods to compute the homology of gaussian groups.

Now rewriting systems quite naturally lead to n-categories, as follows. Let M be a monoid presented by a system (Σ, R) of generators and rewrite rules. If Σ^* denotes the set of words on the alphabet Σ, $R \subset \Sigma^* \times \Sigma^*$ is a set of ordered pairs of words. A rewrite rule $\zeta : x \to y$ applies to any word uxv with $u, v \in \Sigma^*$, defining a reduction step $u\zeta v : uxv \to uyv$. Thus R generates a set R^* of *reduction paths* between words, whose elements are composable sequences of one-step reductions, up to suitable commutation rules (see [14] for a detailed survey). These data fit together in a 2-category

$$\top \leftleftarrows \Sigma^* \leftleftarrows R^*$$

where $\top$ denotes the singleton. It has a unique object, words as arrows and reduction paths as 2-arrows. Here $\leftleftarrows$ denotes the source and target maps: all words clearly have the same source and target, namely the single element of $\top$, and a reduction path from w to w' has of course source w and target w'. Words compose by concatenation, while reduction paths are subject to *two* sorts of composition, either "parallel" or "sequential". What we get exactly is a free 2-category generated by a *computad* [22].

At the next dimension, consider a set $P \subset R^* \times R^*$ of pairs of *parallel* reduction paths, i.e. with the same source and the same target. The smallest equivalence relation on R^* containing P and passing to the context is the *2-congruence generated* by P. In case the relation of parallelism itself is generated by a finite set D, we say that the underlying monoid M is of *finite*

☆ This work has been partially supported by project INVAL (*Invariants algébriques des systèmes informatiques*) ANR-05-BLAN-0267.

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0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2008.10.005

derivation type. It turns out that the latter property holds for all monoids presented by finite, confluent and terminating rewriting systems [20,11]. In n-categorical language, P generates a set P^* of 3-arrows extending the above 2-category to a 3-category:

$$\top \leftleftarrows \Sigma^* \leftleftarrows R^* \leftleftarrows P^*.$$

Note that there are now three ways of composing the elements of P^*. We look here for sets P such that each pair (x, y) of parallel paths in R^* can be filled by at least one $u : x \to y$ in P^*.

This point of view was systematized by the second author [17]. Objects of study are now arbitrary ω-categories, not just monoids; $(\top, \Sigma, R, D)$ becomes an infinite sequence $(S_0, S_1, \ldots, S_n, \ldots)$ defining *n-computads* [19] or *n-polygraphs* [2], a terminology we shall adopt here.

An ω-polygraph, or simply *polygraph* S, generates a free ω-category S^*, generalizing the above situation. There is an abelianization functor taking each polygraph S to a chain-complex $\mathbb{Z}S$ of abelian groups, thus defining a homology

$$\mathrm{H}_*(S) =_{\mathrm{def}} \mathrm{H}_*(\mathbb{Z}S). \tag{1}$$

Now let C be an ω-category, and S a polygraph. A *polygraphic resolution of* C *by* S is a morphism $S^* \to C$ satisfying some lifting properties (see Section 2.3). But the homology $\mathrm{H}_*(S)$ only depends on C [17], so that we may define a "polygraphic homology" of C by

$$\mathrm{H}^{\mathrm{pol}}_*(C) =_{\mathrm{def}} \mathrm{H}_*(S). \tag{2}$$

A monoid M can be seen as a particular ω-category, with degenerate cells but in dimension 1. Thus, for $C = M$, (2) defines the polygraphic homology of M, whence an immediate question:

does $\mathrm{H}^{\mathrm{pol}}_*(M)$ coincide with the usual homology of M, defined by means of resolutions of $\mathbb{Z}$ by free $\mathbb{Z}M$-modules?

It turns out that the answer is positive. The goal of the present article is to present a proof of this result, previously established in the particular case of groups by the first author [13].

The key notion is that of an *unfolding*, an ω-category built upon a polygraphic resolution $S^* \to M$ and from which we recover the usual homology of M by abelianization. This is exposed in Section 3, which contains the core of the argument. As the properties of these unfoldings are heavily based on the results of [17], the paper starts by recalling those results (Section 2); they are however significantly revisited in the following aspects:

- the notion of polygraphic resolution now fits in a Quillen model structure on higher categories [15], generalizing [8,10] (see also [23,24]), whence a new terminology, e.g. *acyclic fibration*;
- the path construction is much simplified (Section 4);
- whereas the results of [17] are sufficient to settle the case of groups, more general statements about homotopy, and new proofs, are needed in the case of arbitrary monoids (Section 5).

This work is part of a general program aiming at a homotopical theory of computations, whose further developments include

- a general finiteness conjecture [14]: is it true that a monoid M presented by a finite, terminating and confluent rewriting system always has a polygraphic resolution $S^* \to M$ where S_i is finite in each dimension?
- the study of other structures expressible by polygraphs, as proof systems [6], Petri nets [7] and term algebras [16]. In the last case, the polygraphic homology is likely to be degenerate; however, resolutions still bear many relevant informations and could lead to new, refined, invariants;
- potential applications to the theory of directed homotopy. See [4] for a survey.

2. Polygraphic homology

2.1. Globular sets and higher categories

Definition 1. A *globular set* is an infinite sequence $S : S_0 \leftleftarrows S_1 \leftleftarrows S_2 \cdots S_i \leftleftarrows S_{i+1} \cdots$, where $S_i \leftleftarrows S_{i+1}$ stands for the *source map* $S_i \overset{\sigma_i}{\leftarrow} S_{i+1}$ and the *target map* $S_i \overset{\tau_i}{\leftarrow} S_{i+1}$, which satisfies the *boundary conditions* $\sigma_i \circ \sigma_{i+1} = \sigma_i \circ \tau_{i+1}$ and $\tau_i \circ \sigma_{i+1} = \tau_i \circ \tau_{i+1}$ for all i. The elements of S_i are called *i-cells*.

We introduce the following notations:

- if x, y are i-cells, we write $x \parallel y$ whenever $i = 0$, or $i > 0$, $\sigma_{i-1}(x) = \sigma_{i-1}(y)$ and $\tau_{i-1}(x) = \tau_{i-1}(y)$;
- if u is an $i+1$-cell, we write $u : x \to y$ whenever $\sigma_i(u) = x$ and $\tau_i(u) = y$, so that x, y are i-cells and $x \parallel y$.

$$\begin{tikzcd} \cdot \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow u"] & \cdot \end{tikzcd}$$

For $j > i$, we introduce the following notations, where $\sigma_{i,j} = \sigma_i \circ \sigma_{i+1} \circ \cdots \circ \sigma_{j-1}$ and $\tau_{i,j} = \tau_i \circ \tau_{i+1} \circ \cdots \circ \tau_{j-1}$:

- if u is a j-cell, we write $u : x \to_i y$ whenever $\sigma_{i,j}(u) = x$ and $\tau_{i,j}(u) = y$, so that x, y are i-cells and $x \parallel y$;
- if u, v are j-cells, we write $u \rhd_i v$ whenever $\tau_{i,j}(u) = \sigma_{i,j}(v)$.

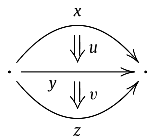
In particular, if u is an i-cell with $i > 0$, we get $u : u^\flat \to_0 u^\sharp$, where $u^\flat$ stands for $\sigma_{0,i}(u)$ and $x^\sharp$ for $\tau_{0,i}(u)$.

Definition 2. If S, T are globular sets, a *homomorphism* $f : S \to T$ is an infinite sequence of maps $f_i : S_i \to T_i$ such that we have $f_{i+1}(u) : f_i(x) \to f_i(y)$ in T for all i and for any $i+1$-cell $u : x \to y$ in S.

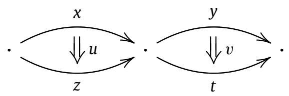
Definition 3. A (strict) *ω-category* is a globular set $C : C_0 \leftleftarrows C_1 \leftleftarrows C_2 \cdots C_i \leftleftarrows C_{i+1} \cdots$ together with *compositions* and *units*, satisfying the laws of *associativity*, *units*, and *interchange*.

In other words, we get:

- some $i+1$-cell $u *_i v : x \to z$ for any $i+1$-cells $u : x \to y$ and $v : y \to z$ (so that $u \rhd_i v$);



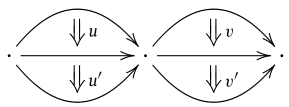
- some $j+1$-cell $u *_i v : x *_i y \to z *_i t$ for all $j > i$ and for any $j+1$-cells $u : x \to z$ and $v : y \to t$ such that $u \rhd_i v$ (so that $x \rhd_i y$ and $z \rhd_i t$);



- some $i+1$-cell $1_x : x \to x$ for any i-cell x. We also write 1_x^{i+1} for this unit.

By induction on $j > i$, we define the $j+1$-cell $1_x^{j+1} : 1_x^j \to 1_x^j$ by $1_x^{j+1} = 1_{1_x^j}$ for any i-cell x. The laws are:

- $(u *_i v) *_i w = u *_i (v *_i w)$ for all $j > i$ and for any j-cells $u \rhd_i v \rhd_i w$;
- $1_x^j *_i u = u = u *_i 1_y^j$ for all $j > i$ and for any j-cell $u : x \to_i y$;
- $(u *_j u') *_i (v *_j v') = (u *_i v) *_j (u' *_i v')$ for all $k > j > i$ and for any k-cells u, u', v, v' such that $u \rhd_i v$, $u \rhd_j u'$ and $v \rhd_j v'$ (so that $u' \rhd_i v'$);



- $1_x *_i 1_y = 1_{x *_i y}$ for all $j > i$ and for any j-cells x, y such that $x \rhd_i y$.

By induction on $k > j > i$, we also get $1_x^k *_i 1_y^k = 1_{x *_i y}^k$ for any j-cells x, y such that $x \rhd_i y$.

By restricting this definition to a finite sequence $C_0 \leftleftarrows C_1 \leftleftarrows C_2 \cdots C_{n-1} \leftleftarrows C_n$, we get the notion of *$n$-category*. Conversely, any n-category is converted into an ω-category by concatenating with the infinite stationary sequence $C_n \leftleftarrows C_n \leftleftarrows C_n \cdots$ where $\sigma_i = \tau_i = \mathrm{id}_{C_n}$ for all $i \geq n$.

In particular, we get the following examples, where $\top$ stands for the singleton set:

- a *set* $S : S \leftleftarrows S \leftleftarrows S \cdots$
- a *monoid* $M : \top \leftleftarrows M \leftleftarrows M \leftleftarrows M \cdots$
- a *category* $C : C_0 \leftleftarrows C_1 \leftleftarrows C_1 \leftleftarrows C_1 \cdots$
- an *abelian monoid* $A : \top \leftleftarrows \top \leftleftarrows A \leftleftarrows A \leftleftarrows A \cdots$
- a *strict monoidal category* $C : \top \leftleftarrows C_1 \leftleftarrows C_2 \leftleftarrows C_2 \leftleftarrows C_2 \cdots$
- a *2-category* $C : C_0 \leftleftarrows C_1 \leftleftarrows C_2 \leftleftarrows C_2 \leftleftarrows C_2 \cdots$.

Note that we use the same notation for a monoid M, its underlying set, and its associated ω-category.

An ω-category (respectively an n-category) such that $C_0 = \top$ is called an *ω-monoid* (respectively an *n-monoid*). In that case, $x \rhd_0 y$ holds for any $x, y \in C_i$ with $i > 0$. So we write xy for $x *_0 y$, and 1 for the single unit 1-cell.

Definition 4. If C and D are ω-categories, an *ω-functor* $f : C \to D$ is a homomorphism such that each map $f_i : C_i \to D_i$ is compatible with compositions and units. In other words, the following conditions hold:

- $f_j(x *_i y) = f_j(x) *_i f_j(y)$ for all $j > i$ and for any j-cells $x \triangleright_i y$ in C;
- $f_{i+1}(1_x) = 1_{f_i(x)}$ for any i-cell x in C.

So we get a *category of ω-categories*. Note that this category has all limits, which are defined in the obvious way. In particular, the terminal object is the *trivial ω-category* $\top : \top \leftleftarrows \top \leftleftarrows \top \cdots$.

Note also that, in the case where C is an ω-monoid and M is a monoid, an ω-functor $f : C \to M$ is completely given by a map $f_1 : C_1 \to M$ satisfying the following three conditions:

$$f_1(xy) = f_1(x)f_1(y) \quad \text{for any 1-cells } x, y \text{ in } C, f_1(1) = 1, \qquad f_1(x) = f_1(y) \quad \text{for any 2-cell } u : x \to y \text{ in } C.$$

Indeed, we have $f_i = f_1 \circ \sigma_{1,i} = f_1 \circ \tau_{1,i}$ for all $i > 1$, and all conditions are consequences of the above three.

2.2. Polygraphs

A *graph* $S_0 \leftleftarrows S_1$ consists of sets S_0, S_1 and maps $S_0 \xleftarrow{\sigma_0} S_1$ and $S_0 \xleftarrow{\tau_0} S_1$. It generates a *free category* $S_0 \leftleftarrows S_1^*$, where S_1^* is the set of paths in the graph $S_0 \leftleftarrows S_1$.

Similarly, if $n > 0$ and $C_0 \leftleftarrows C_1 \leftleftarrows C_2 \cdots C_{n-1} \leftleftarrows C_n$ is an n-category, then any graph $C_n \leftleftarrows S_{n+1}$ satisfying the boundary conditions $\sigma_{n-1} \circ \sigma_n = \sigma_{n-1} \circ \tau_n$ and $\tau_{n-1} \circ \sigma_n = \tau_{n-1} \circ \tau_n$ *freely generates* the $n+1$-category $C_0 \leftleftarrows C_1 \leftleftarrows C_2 \cdots C_{n-1} \leftleftarrows C_n \leftleftarrows S_{n+1}^*$, where S_{n+1}^* consists of formal compositions of elements of S_{n+1}. Hence, the latter are called *$n+1$-generators*. See [2,18] for a detailed construction of S_{n+1}^*.

Definition 5 (*[2]*). The notion of *n-polygraph* is defined by induction on $n > 0$:

- a *1-polygraph* is a graph $S_0^* \leftleftarrows S_1$, where S_0^* is just another notation for the set S_0;
- if $n > 0$, an *$n+1$-polygraph* is given by an n-polygraph $S_0^* \leftleftarrows S_1, S_1^* \leftleftarrows S_2, \ldots, S_{n-1}^* \leftleftarrows S_n$ together with a graph $S_n^* \leftleftarrows S_{n+1}$ satisfying the boundary conditions $\sigma_{n-1} \circ \sigma_n = \sigma_{n-1} \circ \tau_n$ and $\tau_{n-1} \circ \sigma_n = \tau_{n-1} \circ \tau_n$. It generates the *free $n+1$-category* $S^* : S_0^* \leftleftarrows S_1^* \leftleftarrows S_2^* \cdots S_n^* \leftleftarrows S_{n+1}^*$.

Polygraphs are equivalent to *computads*: See [22,19]. Here are two basic cases:

- an *alphabet* $S_1 = \{\xi_1, \xi_2, \ldots\}$ yields a graph $\top \leftleftarrows S_1$ with only one vertex. The free category generated by this graph is $\top \leftleftarrows S_1^*$, where S_1^* is the free monoid generated by S_1;
- a *rewriting system* on S_1^*, given by the set of rules $S_2 = \{x_1 \xrightarrow{\zeta_1} y_1, x_2 \xrightarrow{\zeta_2} y_2, \ldots\}$, defines a graph $S_1^* \leftleftarrows S_2$. We get a 2-polygraph, since the boundary conditions are trivially satisfied, and the free 2-category generated by this 2-polygraph is the 2-monoid $\top \leftleftarrows S_1^* \leftleftarrows S_2^*$, where S_2^* is the set of reductions modulo interchange.

Therefore, an n-polygraph can be considered as a higher dimensional rewriting system (*syntactical interpretation*). It can also be seen as a kind of directed CW-complex (*geometric interpretation*). Various examples of 3-polygraphs corresponding to higher dimensional rewriting systems are given in [12]. See also [5–7].

Definition 6 (*[2]*). A *polygraph* is an infinite sequence $S_0^* \leftleftarrows S_1, S_1^* \leftleftarrows S_2, \ldots, S_i^* \leftleftarrows S_{i+1}, \ldots$ whose first items define an i-polygraph for all $i > 0$. It generates the *free ω-category* $S^* : S_0^* \leftleftarrows S_1^* \leftleftarrows S_2^* \cdots S_i^* \leftleftarrows S_{i+1}^* \cdots$.

Note that the trivial ω-category $\top$ coincides with the free ω-category Ω^*, where $\Omega_0 = \top$ and $\Omega_i = \emptyset$ for all $i > 0$.

2.3. Polygraphic resolutions

Definition 7. An ω-functor $p : C \to D$ is an *acyclic fibration* if $p_0 : C_0 \to D_0$ is onto and p has the *lifting property*: For any i-cells $x \parallel y$ in C and for any $v : p_i(x) \to p_i(y)$ in D, there is some $u : x \to y$ in C such that $p_{i+1}(u) = v$.

$$\begin{tikzcd} \cdot \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow u"] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, bend left, "p_i(x)"] \arrow[r, bend right, "p_i(y)"'] \arrow[r, phantom, "\Downarrow v"] & \cdot \end{tikzcd}$$

Note that if $p : C \to D$ is an acyclic fibration, then each $p_i : C_i \to D_i$ is onto and p has the *stretching property*: For any i-cells $x \parallel y$ in C such that $p_i(x) = p_i(y) = z$ in D, there is some $u : x \to y$ in C such that $p_{i+1}(u) = 1_z$.

$$\begin{tikzcd} \cdot \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow u"] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "z"] & \cdot \end{tikzcd}$$

Conversely, those properties characterize acyclic fibrations: See [17].

Note also that our acyclic fibrations are the *trivial fibrations* of some model structure. See [15].

Definition 8. We say that an ω-category C is *acyclic* if the canonical ω-functor $\pi : C \to \top$ is an acyclic fibration. In other words, C_0 is inhabited and C has the *filling property*: For any i-cells $x \parallel y$ in C, there is some $u : x \to y$.

Proposition 1 (*[17]*). *For any acyclic fibration $p : C \to D$ and for any $q : S^* \to D$, there is some $f : S^* \to C$ such that $q = p \circ f$.*

$$\begin{tikzcd} & C \arrow[d, "p"] \\ S^* \arrow[ur, dashed, "f"] \arrow[r, "q"'] & D \end{tikzcd}$$

In other words, free ω-categories are *cofibrant*. It suffices indeed to define the i-cell $f_i(\xi)$ for each i-generator ξ, using the fact that p is an acyclic fibration. In fact, the converse holds: Cofibrant ω-categories are free [18].

Proposition 2 (*[17]*). *For any $p : C \to D$ and $f, g : S^* \to C$ such that $p \circ f = p \circ g$ and p has the lifting property, we get a* homotopy *$f \rightsquigarrow g$.*

The definition of homotopy and the proof of this result are postponed to Section 5.

Definition 9 (*[17]*). A *polygraphic resolution* of C is an acyclic fibration $p : S^* \to C$ where S^* is free.

Theorem 1 (*[17]*).

1. *Any ω-category C has a polygraphic resolution $p : S^* \to C$.*
2. *If $p : S^* \to C$ and $q : T^* \to C$ are polygraphic resolutions, there is some $f : S^* \to T^*$ such that $p = q \circ f$.*

$$\begin{tikzcd} S^* \arrow[rr, dashed, "f"] \arrow[dr, "p"'] & & T^* \arrow[dl, "q"] \\ & C & \end{tikzcd}$$

3. *For any two such $f, g : S^* \to T^*$, we get a homotopy $f \rightsquigarrow g$.*

Proof. We build S_i and p_i by induction on i, starting from $S_0 = C_0$ and $p_0 = \mathrm{id}_{C_0}$: For any $x, y \in S_i^*$ with $x \parallel y$ and for any $i+1$-cell $v : p_i(x) \to p_i(y)$ in C, we introduce some $i+1$-generator $\xi : x \to y$ such that $p_{i+1}(\xi) = v$. By construction, we get a polygraphic resolution $p : S^* \to C$. The rest follows from Propositions 1 and 2. $\square$

Corollary 1. *If $p : S^* \to C$ and $q : T^* \to C$ are polygraphic resolutions, there are $f : S^* \to T^*$ and $g : T^* \to S^*$ such that the following conditions hold:*

$$p = q \circ f, \qquad q = p \circ g, \qquad g \circ f \rightsquigarrow \mathrm{id}_{S^*}, \qquad f \circ g \rightsquigarrow \mathrm{id}_{T^*}.$$

In other words, any two polygraphic resolutions of C are homotopically equivalent.

Note also that any monoid M has a *monoidal resolution*, that is a polygraphic resolution such that $S_0^* = S_0 = \top$. Such a resolution contains a presentation of M, where S_1 is the set of generators and S_2 is the set of relations. Moreover, any such presentation of M is *reversible*: For any reduction $x \to^* y$, there is another reduction $y \to^* x$. Conversely, any reversible presentation of M can be extended to a monoidal resolution of M.

In general, a rewrite system for M is not reversible, but we get a reversible presentation by adding *inverse rules*. The following theorem is conjectured in [14]: If we start from some *finite convergent rewrite system*, then the corresponding reversible presentation extends to a monoidal resolution $p : S^* \to M$ such that all S_i are finite.

2.4. Abelianization and homology

Let $S^* : S_0^* \leftleftarrows S_1^* \leftleftarrows S_2^* \cdots S_i^* \leftleftarrows S_{i+1}^* \cdots$ be a free ω-category. If ξ is an i-generator, we write $\lceil \xi \rfloor$ for the corresponding generator of the free $\mathbb{Z}$-module $\mathbb{Z}S_i$, and we extend this notation to all cells of S^* as follows:

$$\lceil u *_i v \rfloor = \lceil u \rfloor + \lceil v \rfloor \quad \text{for any } j\text{-cells } u \rhd_i v \text{ in } S^* \text{ with } j > i, \qquad \lceil 1_x \rfloor = 0 \quad \text{for any } i\text{-cell } x \text{ in } S^*.$$

In other words, $\lceil x \rfloor$ counts the number of occurrences of each i-generator in the i-cell x. The fact that $\lceil x \rfloor$ is well defined follows from the universal property of S_i^* and the definition of some suitable i-category: See Appendix A.

Now we define $\mathbb{Z}$-linear maps $\mathbb{Z}S_i \xleftarrow{\partial_i} \mathbb{Z}S_{i+1}$ as follows: $\partial_i \lceil \xi \rfloor = \lceil y \rfloor - \lceil x \rfloor$ for each $i+1$-generator $\xi : x \to y$.

Lemma 1. $\partial_i \lceil u \rfloor = \lceil y \rfloor - \lceil x \rfloor$ *for any $i+1$-cell $u : x \to y$ in S^*.*

This is easily proved by induction on u. As a consequence, we get $\partial_i \circ \partial_{i+1} = 0$ for all i.

Definition 10 ([17]). The *abelianization of the free ω-category $S^* : S_0^* \leftleftarrows S_1^* \leftleftarrows S_2^* \cdots S_i^* \leftleftarrows S_{i+1}^* \cdots$* is the following chain-complex of free $\mathbb{Z}$-modules:

$$\mathbb{Z}S : \mathbb{Z}S_0 \xleftarrow{\partial_0} \mathbb{Z}S_1 \xleftarrow{\partial_1} \mathbb{Z}S_2 \cdots \mathbb{Z}S_i \xleftarrow{\partial_i} \mathbb{Z}S_{i+1} \cdots.$$

For any $f : S^* \to T^*$, we define $\mathbb{Z}$-linear maps $f_i^{\mathrm{ab}} : \mathbb{Z}S_i \to \mathbb{Z}T_i$ as follows: $f_i^{\mathrm{ab}}\lceil\xi\rfloor = \lceil f_i(\xi)\rfloor$ for each $\xi \in S_i$.

Lemma 2. *$f_i^{\mathrm{ab}}\lceil x\rfloor = \lceil f_i(x)\rfloor$ for any $x \in S_i^*$.*

This is easily proved by induction on x. As a consequence, we get $\partial_i \circ f_{i+1}^{\mathrm{ab}} = f_i^{\mathrm{ab}} \circ \partial_i$ for all i.

Definition 11 ([17]). The *abelianization of the ω-functor $f : S^* \to T^*$* is the homomorphism of chain-complex $f^{\mathrm{ab}} : \mathbb{Z}S \to \mathbb{Z}T$ given by the $f_i^{\mathrm{ab}} : \mathbb{Z}S_i \to \mathbb{Z}T_i$.

Note that abelianization is defined in terms of polygraphs, but in fact, it only depends on the generated ω-categories. Obviously, we have $(g \circ f)^{\mathrm{ab}} = g^{\mathrm{ab}} \circ f^{\mathrm{ab}}$ for any $f : R^* \to S^*$ and $g : S^* \to T^*$, and $\mathrm{id}_{S^*}^{\mathrm{ab}} = \mathrm{id}_{\mathbb{Z}S}$ for any S^*. Hence, we get a functor from the category of free ω-categories to the category of chain-complexes.

Proposition 3 ([17]). *For any $f, g : S^* \to T^*$ such that $f \rightsquigarrow g$, we get a chain-homotopy between f^{ab} and g^{ab}.*

This crucial result is proved in Section 5. By Corollary 1, we get:

Corollary 2. *The homology of $\mathbb{Z}S$ does not depend on the choice of the polygraphic resolution $p : S^* \to C$.*

Definition 12 ([17]). The homology of such a $\mathbb{Z}S$ is called the *polygraphic homology of the ω-category C*.

Note that Ω^* defines a polygraphic resolution of $\top$, and so does any acyclic free ω-category. Hence, we get:

Corollary 3. *The following augmented chain-complex is exact whenever S^* is an acyclic free ω-category:*

$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}S_0 \xleftarrow{\partial_0} \mathbb{Z}S_1 \xleftarrow{\partial_1} \mathbb{Z}S_2 \cdots \mathbb{Z}S_i \xleftarrow{\partial_i} \mathbb{Z}S_{i+1} \cdots.$$

Here, ε stands for π_0^{ab} where $\pi : S^* \to \Omega^* = \top$ is the canonical ω-functor. Hence, $\varepsilon(\xi) = 1$ for each $\xi \in S_0$.

3. Unfolding

If M is a monoid and S is a set, we write $M \cdot S$ for the cartesian product $M \times S$ whose elements are written $\lambda \cdot x$. The *free action* of M on the set $M \cdot S$ is defined by $\lambda \cdot (\mu \cdot x) = \lambda\mu \cdot x$ for all $\lambda, \mu \in M$ and $x \in S$. In particular, we shall identify the set $M \cdot \top$ with M, where the action of M on itself is defined by $\lambda \cdot \mu = \lambda\mu$ for all $\lambda, \mu \in M$.

Note also that $\mathbb{Z}(M \cdot S)$ has a structure of $\mathbb{Z}M$-module defined by $\lambda \cdot (\mu \cdot x) = \lambda\mu \cdot x$ for any $\lambda, \mu \in M$ and $x \in S$. In particular, we get $\lambda \cdot x = \lambda \cdot (1 \cdot x)$. Hence, we shall identify $\mathbb{Z}(M \cdot S)$ with the free $\mathbb{Z}M$-module $\mathbb{Z}M \cdot S$.

3.1. General case

Let $f : C \to M$ be an ω-functor, where M is a monoid and C is an ω-monoid, so that $M \cdot C_0 = M \cdot \top = M$. If x is an i-cell in C with $i > 0$, we write $\bar{x}$ for $f_i(x) \in M$. In particular, we get $\bar{x} = \bar{y}$ for any 2-cell $u : x \to y$. Moreover, we get $\bar{x} = \bar{y}$ for all $i > 1$ and for any i-cells x, y such that $x \parallel y$.

We define the globular set $M \cdot C : M \leftleftarrows M \cdot C_1 \leftleftarrows M \cdot C_2 \cdots M \cdot C_i \leftleftarrows M \cdot C_{i+1} \cdots$ as follows:

- we get the 1-cell $\lambda \cdot x : \lambda \to \lambda\bar{x}$ in $M \cdot C$ for any $\lambda \in M$ and for any 1-cell x in C;
- if $i > 1$, we get the i-cell $\lambda \cdot u : \lambda \cdot x \to \lambda \cdot y$ in $M \cdot C$ for any $\lambda \in M$ and for any i-cell $u : x \to y$ in C.

As a consequence, we get the following characterization of $\parallel$ in $M \cdot C$:

- for any $\lambda, \mu \in M$ and for any 1-cells x, y in C, we have $\lambda \cdot x \parallel \mu \cdot y$ iff $\lambda = \mu$ and $\lambda\bar{x} = \lambda\bar{y}$;
- for any $\lambda, \mu \in M$ and for any i-cells x, y in C with $i > 1$, we have $\lambda \cdot x \parallel \mu \cdot y$ iff $\lambda = \mu$ and $x \parallel y$.

In particular, for any 2-cell $\lambda \cdot u : \lambda \cdot x \to \lambda \cdot y$ in $M \cdot C$, we have $u : x \to y$ in C, so that $\bar{x} = \bar{y}$ and $\lambda \cdot x \parallel \lambda \cdot y$. The other boundary conditions for $M \cdot C$ follow directly from the boundary conditions for C.

We also get the following characterization of iterated sources and targets in $M \cdot C$:

- if $i > 0$, we get $\lambda \cdot x : \lambda \to_0 \lambda\bar{x}$ in $M \cdot C$ for any $\lambda \in M$ and for any i-cell x in C;
- if $j > i > 0$, we get $\lambda \cdot u : \lambda \cdot x \to_i \lambda \cdot y$ in $M \cdot C$ for any $\lambda \in M$ and for any j-cell $u : x \to_i y$ in C.

As a consequence, we get the following characterization of $\triangleright_i$ in $M \cdot C$:

- for any $\lambda, \mu \in M$ and for any i-cells x, y in C with $i > 0$, we have $\lambda \cdot x \triangleright_0 \mu \cdot y$ iff $\lambda\overline{x} = \mu$;
- for any $\lambda, \mu \in M$ and for any j-cells x, y in C with $j > i > 0$, we have $\lambda \cdot x \triangleright_i \mu \cdot y$ iff $\lambda = \mu$ and $x \triangleright_i y$.

Using this, we define compositions and units in $M \cdot C$ as follows:

- $(\lambda \cdot x) *_0 (\lambda\overline{x} \cdot y) = \lambda \cdot xy$ for any $\lambda \in M$ and for any i-cells x, y in C with $i > 0$;
- $(\lambda \cdot x) *_i (\lambda \cdot y) = \lambda \cdot (x *_i y)$ for any $\lambda \in M$ and for any j-cells $x \triangleright_i y$ in C with $j > i > 0$;
- $1_{\lambda \cdot x} = \lambda \cdot 1_x$ for any $\lambda \in M$ and for any i-cell x in C. In particular, $1_\lambda = \lambda \cdot 1$ for any $\lambda \in M$.

It is easy to see that those operations satisfy the laws of associativity, left and right unit, and interchange. Moreover, we have an obvious ω-functor $\tilde{f} : M \cdot C \to C$ defined by $\tilde{f}_i(\lambda \cdot x) = x$ for any $\lambda \in M$ and $x \in C_i$.

Definition 13. The ω-category $M \cdot C : M \leftleftarrows M \cdot C_1 \leftleftarrows M \cdot C_2 \cdots M \cdot C_i \leftleftarrows M \cdot C_{i+1} \cdots$ defined as above is called the *unfolding of $f : C \to M$*, and the ω-functor $\tilde{f} : M \cdot C \to C$ is called its *folding ω-functor*.

Note that the action of M on $M \cdot C$ is compatible with this structure of ω-category.

3.2. Unfolding an acyclic fibration

Proposition 4. *If G is a group, then the unfolding $G \cdot C$ of an acyclic fibration $p : C \to G$ is an acyclic ω-category.*

Proof. G is inhabited, and using the fact that p is an acyclic fibration, we prove the filling property for $G \cdot C$:

- if $\lambda, \mu \in G$, there is some $x \in C_1$ such that $\overline{x} = \lambda^{-1}\mu$, and we get $\lambda \cdot x : \lambda \to \lambda\overline{x} = \lambda\lambda^{-1}\mu = \mu$ in $G \cdot C$;
- if $\lambda \cdot x \parallel \mu \cdot y$ where $\lambda, \mu \in G$ and $x, y \in C_1$, we get $\lambda = \mu$ and $\lambda\overline{x} = \lambda\overline{y}$, so that $\overline{x} = \overline{y}$ by left cancellation. Hence, there is some 2-cell $u : x \to y$ in C, and we get $\lambda \cdot u : \lambda \cdot x \to \lambda \cdot y = \mu \cdot y$ in $G \cdot C$;
- finally, if $i > 1$ and $\lambda \cdot x \parallel \mu \cdot y$ where $\lambda, \mu \in G$ and $x, y \in C_i$, we get $\lambda = \mu$ and $x \parallel y$, so that $\overline{x} = \overline{y}$. Hence, there is some $i + 1$-cell $u : x \to y$ in C, and we get $\lambda \cdot u : \lambda \cdot x \to \lambda \cdot y = \mu \cdot y$ in $G \cdot C$. $\square$

In fact, the converse holds: If the unfolding $M \cdot C$ of $f : C \to M$ is an acyclic ω-category, then M is a group and f is an acyclic fibration. Hence, the above result cannot hold for arbitrary monoids, but we have a weaker result:

Proposition 5. *The unfolding $M \cdot C$ of an acyclic fibration $p : C \to M$ has the following* relative filling property*:*

- *for any $\mu \in M$, there is some $1 \cdot x : 1 \to \mu$ in $M \cdot C$;*
- *for any $x, y \in C_i$ with $i > 0$ such that $1 \cdot x \parallel 1 \cdot y$, there is some $1 \cdot u : 1 \cdot x \to 1 \cdot y$ in $M \cdot C$.*

No extra assumption on the monoid M is needed here, since $\lambda = 1$ has a right inverse and is left cancellable.

3.3. Free case

Now we consider $f : S^* \to M$, where S^* is a free ω-monoid. Hence, $S_0^* = S_0 = \top$ and $M \cdot S_0^* = M \cdot \top = M$. We shall see that the unfolding $M \cdot S^* : M \leftleftarrows M \cdot S_1^* \leftleftarrows M \cdot S_2^* \cdots M \cdot S_i^* \leftleftarrows M \cdot S_{i+1}^* \cdots$ is a free ω-category.

If $n > 0$, we have a canonical injection of $M \cdot S_n$ into $M \cdot S_n^*$, from which we get a graph $M \cdot S_{n-1}^* \leftleftarrows M \cdot S_n$, and if $n > 1$, the boundary conditions $\sigma_{n-2} \circ \sigma_{n-1} = \sigma_{n-2} \circ \tau_{n-1}$ and $\tau_{n-2} \circ \sigma_{n-1} = \tau_{n-2} \circ \tau_{n-1}$ are satisfied. We get the n-category $M \leftleftarrows M \cdot S_1^* \leftleftarrows M \cdot S_2^* \cdots M \cdot S_{n-2}^* \leftleftarrows M \cdot S_{n-1}^* \leftleftarrows (M \cdot S_n)^*$, and the canonical injection extends to $\varphi_n : (M \cdot S_n)^* \to M \cdot S_n^*$, which is compatible with sources, targets, compositions and units.

If $\lambda \in M$ and $\xi \in S_n$, we get $\lambda \cdot \xi \in M \cdot S_n$ and we write $\langle \lambda \cdot \xi \rangle$ for the corresponding element of $(M \cdot S_n)^*$. More generally, if $\lambda \in M$ and $x \in S_n^*$, we get $\lambda \cdot x \in M \cdot S_n^*$ and we define $\langle \lambda \cdot x \rangle \in (M \cdot S_n)^*$ by induction on x, in such a way that $\langle \lambda \cdot x \rangle$ has the same source and the same target as $\lambda \cdot x$:

- $\langle \lambda \cdot xy \rangle = \langle \lambda \cdot x \rangle *_0 \langle \lambda\overline{x} \cdot y \rangle$ for any $\lambda \in M$ and for any n-cells x, y in S^*;
- $\langle \lambda \cdot (x *_i y) \rangle = \langle \lambda \cdot x \rangle *_i \langle \lambda \cdot y \rangle$ for any $\lambda \in M$ and for any n-cells $x \triangleright_i y$ in S^* with $n > i > 0$;
- $\langle \lambda \cdot 1_x \rangle = 1_{\lambda \cdot x}$ for any $\lambda \in M$ and for any $n - 1$-cell x in S^*. In particular, $\langle \lambda \cdot 1 \rangle = 1_\lambda \in (M \cdot S_1)^*$.

In other words, $\langle \lambda \cdot x \rangle$ is a decomposition of the cell $\lambda \cdot x$ into elements of $M \cdot S_n$. The fact that it is well defined follows from the universal property of S_n^* and from the definition of some suitable n-category: See Appendix B. Hence, we get $\psi_n : M \cdot S_n^* \to (M \cdot S_n)^*$, which is compatible with sources, targets, compositions and units.

By construction, we have $\varphi_n \langle \lambda \cdot \xi \rangle = \lambda \cdot \xi$ for any $\lambda \in M$ and $\xi \in S_n$, so that $\psi_n(\varphi_n \langle \lambda \cdot \xi \rangle) = \psi_n(\lambda \cdot \xi) = \langle \lambda \cdot \xi \rangle$. By the universal property of $(M \cdot S_n)^*$, the map $\psi_n \circ \varphi_n$ is the identity on $(M \cdot S_n)^*$.

Lemma 3. *$\varphi_n \langle \lambda \cdot x \rangle = \lambda \cdot x$ for any $\lambda \in M$ and $x \in S_n^*$. In other words, $\varphi_n \circ \psi_n$ is the identity on $M \cdot S_n^*$.*

This is easily proved by induction on x. So we can identify $M \cdot S_n^*$ with $(M \cdot S_n)^*$ and we get the following result:

Proposition 6. *The unfolding* $M\cdot S^* : M \leftleftarrows M\cdot S_1^* \leftleftarrows M\cdot S_2^* \cdots M\cdot S_i^* \leftleftarrows M\cdot S_{i+1}^* \cdots$ *can be identified with a free* ω*-category* $(M\cdot S)^* : M \leftleftarrows (M\cdot S_1)^* \leftleftarrows (M\cdot S_2)^* \cdots (M\cdot S_i)^* \leftleftarrows (M\cdot S_{i+1})^* \cdots$.

By abelianization of $M\cdot S^* = (M\cdot S)^*$, we get the following chain-complex of free $\mathbb{Z}$-modules:

$$\mathbb{Z}(M\cdot S) : \mathbb{Z}M \overset{\partial_0}{\leftarrow} \mathbb{Z}(M\cdot S_1) \overset{\partial_1}{\leftarrow} \mathbb{Z}(M\cdot S_2)\cdots \mathbb{Z}(M\cdot S_i) \overset{\partial_i}{\leftarrow} \mathbb{Z}(M\cdot S_{i+1})\cdots.$$

Moreover, $\mathbb{Z}(M\cdot S_i)$ can be identified with the free $\mathbb{Z}M$-module $\mathbb{Z}M\cdot S_i$.

Lemma 4. $\lceil \lambda\cdot x\rfloor = \lambda\cdot\lceil 1\cdot x\rfloor$ *for any* $\lambda\in M$ *and* $x\in S_i^*$.

This is proved by induction on x. As a consequence, we get $\partial_i(\lambda\cdot\xi) = \lambda\cdot\partial_i(1\cdot\xi)$ for any $\lambda\in M$ and $\xi\in S_{i+1}$. In other words, $\partial_i : \mathbb{Z}M\cdot S_{i+1} \to \mathbb{Z}M\cdot S_i$ is $\mathbb{Z}M$-linear.

We also get $\tilde{f}_i^{\mathrm{ab}}\lceil\lambda\cdot\xi\rfloor = \lceil\xi\rfloor$ for any $\lambda\in M$ and $\xi\in S_i$. In other words, $\tilde{f}_i^{\mathrm{ab}} : \mathbb{Z}M\cdot S_i \to \mathbb{Z}S_i$ is $\mathbb{Z}M$-linear if we consider the *trivial action* of M on $\mathbb{Z}S_i$.

To sum up, we get the following result:

Proposition 7. *The abelianization of the unfolding* $M\cdot S^* = (M\cdot S)^*$ *yields a chain-complex of free* $\mathbb{Z}M$*-modules:*

$$\mathbb{Z}M\cdot S : \mathbb{Z}M \overset{\partial_0}{\leftarrow} \mathbb{Z}M\cdot S_1 \overset{\partial_1}{\leftarrow} \mathbb{Z}M\cdot S_2\cdots \mathbb{Z}M\cdot S_i \overset{\partial_i}{\leftarrow} \mathbb{Z}M\cdot S_{i+1}\cdots.$$

Furthermore, the chain-complex $\mathbb{Z}S$ *is obtained by trivializing the action of* M *in* $\mathbb{Z}M\cdot S$.

3.4. Unfolding a resolution

Now we can state our main result:

Theorem 2. *The unfolding of a monoidal resolution* $p : S^* \to M$ *yields is a resolution of* $\mathbb{Z}$ *by free* $\mathbb{Z}M$*-modules:*

$$0 \leftarrow \mathbb{Z} \overset{\varepsilon}{\leftarrow} \mathbb{Z}M \overset{\partial_0}{\leftarrow} \mathbb{Z}M\cdot S_1 \overset{\partial_1}{\leftarrow} \mathbb{Z}M\cdot S_2\cdots \mathbb{Z}M\cdot S_i \overset{\partial_i}{\leftarrow} \mathbb{Z}M\cdot S_{i+1}\cdots.$$

Here, ε is defined by $\varepsilon(\lambda) = 1$ for all $\lambda\in M$. It is $\mathbb{Z}M$-linear if we consider the trivial action of M on $\mathbb{Z}$.

Since the homology of M is obtained by trivializing the action of M in such a resolution, we get:

Corollary 4. *The homology of a monoid M coincides with its polygraphic homology.*

For groups, Theorem 2 follows from Proposition 4 and Corollary 3.

For monoids, we need a little more. First, we consider some ω-category C and two subsets $\mathcal{X}, \mathcal{Y} \subset C_0$.

Definition 14. $p : C \to D$ has the *lifting property with respect to* $(\mathcal{X}, \mathcal{Y})$ if the following conditions hold:

- for any $x\in\mathcal{X}$, $y\in\mathcal{Y}$, and $v : p_0(x)\to p_0(y)$ in D, there is some $u : x\to y$ in C such that $p_1(u) = v$;

$$\mathcal{X}\ni x \xrightarrow{u} y\in\mathcal{Y} \qquad\qquad p_0(x)\xrightarrow{v} p_0(y)$$

- for any i-cells $x \parallel y$ in C with $i > 0$ such that $x^\flat\in\mathcal{X}$ and $x^\sharp\in\mathcal{Y}$, and for any $v : p_i(x)\to p_i(y)$ in D, there is some $u : x\to y$ in C such that $p_{i+1}(u) = v$.

$$\begin{tikzcd} \mathcal{X}\ni x^\flat \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow u"] & x^\sharp\in\mathcal{Y} \end{tikzcd} \qquad\qquad \begin{tikzcd} p_i(x^\flat) \arrow[r, bend left, "p_i(x)"] \arrow[r, bend right, "p_i(y)"'] \arrow[r, phantom, "\Downarrow v"] & p_i(x^\sharp) \end{tikzcd}.$$

If $\mathcal{X} = \mathcal{Y}$, we say that p has the *lifting property with respect to* $\mathcal{X}$.

Thus the lifting property with respect to $\mathcal{X} = C_0$ is just the lifting property of Definition 7.

Note that there is a straightforward generalization of Proposition 1:

Proposition 8. *Let* $p : C\to D$, $q : S^*\to D$ *and* $\mathcal{X}\subset C_0$. *If* $q_0(S_0^*)\subset p_0(\mathcal{X})$ *and p has the lifting property with respect to* $\mathcal{X}$, *then there is an* $f : S^*\to C$ *such that* $q = p\circ f$.

Using this, the following generalization of Proposition 2 is proved in Section 5:

Proposition 9. *For any* $p : C\to D$ *and* $f, g : S^*\to C$ *such that* $p\circ f = p\circ g$ *and p satisfies the lifting property with respect to* $\big(f_0(S_0^*), g_0(S_0^*)\big)$, *we get a homotopy* $f \rightsquigarrow g$.

Now we consider the unfolding $C = M\cdot S^* = (M\cdot S)^*$ of some monoidal resolution of M, so that $C_0 = M$.

In that case, we have two canonical ω-functors $\pi : C\to\top$, and $\iota : \top\to C$ corresponding to the 0-cell $1\in M$. For $f = \iota\circ\pi : C\to C$ and $g = \mathrm{id}_C$, we get $\pi\circ f = \pi = \pi\circ g$, $f_0(S_0^*) = \{1\}$ and $g_0(S_0^*) = M$. By Proposition 5, $\pi : C\to\top$ has the lifting property with respect to $(\{1\}, M)$. Hence, we get a homotopy $\iota\circ\pi \rightsquigarrow \mathrm{id}_C$.

By Proposition 3, the augmented chain-complex of Theorem 2 is exact and we are done.

4. Path ω-category

Let C be an ω-category. For any 0-cells x, y in C, we define the ω-category $[x, y]$ as follows:

- there is an i-cell $[u]$ in $[x, y]$ for each $i + 1$-cell $u : x \to_0 y$ in C;
- we get $[w] : [u] \to [v]$ in $[x, y]$ for any $i + 1$-cells $u, v : x \to_0 y$ and for any $i + 2$-cell $w : u \to v$ in C;
- compositions are defined by $[u] *_i [v] = [u *_{i+1} v]$ whenever $u \rhd_{i+1} v$, and units by $1_{[u]} = [1_u]$.

If $j > i > 0$, we write $u *_0 v$ for the j-cell $1^j_u *_0 v$ whenever u is an i-cell and v is a j-cell such that $1^j_u \rhd_0 v$ or for the j-cell $u *_0 1^j_v$ whenever u is a j-cell and v is an i-cell such that $u \rhd_0 1^j_v$. For any 0-cells x, y, z, we get:

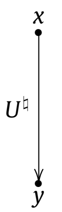
- the *precomposition* ω-functor $u \cdot - : [y, z] \to [x, z]$ for each 1-cell $u : x \to y$, defined by $u \cdot [v] = [u *_0 v]$;
- the *postcomposition* ω-functor $- \cdot v : [x, y] \to [x, z]$ for each 1-cell $v : y \to z$, defined by $[u] \cdot v = [u *_0 v]$;
- the *composition* ω-bifunctor $- \circledast - : [x, y] \times [y, z] \to [x, z]$, defined by $[u] \circledast [v] = [u *_0 v]$.

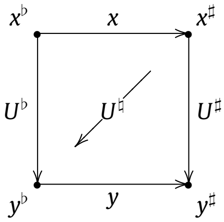
4.1. Cylinders

Definition 15. By induction on i, we define the notion of *i-cylinder $U : x \curvearrowright y$* between i-cells x and y in C:

- a 0-cylinder $U : x \curvearrowright y$ is given by some 1-cell $U^\natural : x \to y$;
- if $i > 0$, an i-cylinder $U : x \curvearrowright y$ is given by two 1-cells $U^\flat : x^\flat \to y^\flat$ and $U^\sharp : x^\sharp \to y^\sharp$, together with some $i - 1$-cylinder $[U] : [x] \cdot U^\sharp \curvearrowright U^\flat \cdot [y]$ in the ω-category $[x^\flat, y^\sharp]$.

If $U : x \curvearrowright y$ is such an i-cylinder, we write $\pi^1 U$ for its *top cell* x and $\pi^2 U$ for its *bottom cell* y. Finally, we write $U^\natural$ for its *principal cell*, which is inductively defined by $\left[U^\natural\right] = [U]^\natural$: It is an $i + 1$-cell in C.

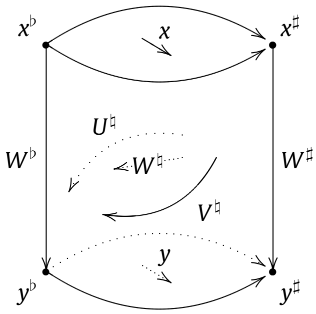




Definition 16. By induction on i, we define the *source i-cylinder $U : x \curvearrowright x'$* and the *target i-cylinder $V : y \curvearrowright y'$* of any $i + 1$-cylinder $W : z \curvearrowright z'$ between $i + 1$-cells $z : x \to y$ and $z' : x' \to y'$ in C:

- if $i = 0$, then $U^\natural = W^\flat$ and $V^\natural = W^\sharp$;
- if $i > 0$, then $U^\flat = V^\flat = W^\flat$ and $U^\sharp = V^\sharp = W^\sharp$, whereas the two $i - 1$-cylinders $[U]$ and $[V]$ are respectively defined as the source and the target of the i-cylinder $[W]$ in $[z^\flat, z'^\sharp]$.

In that case, we write $W : U \to V$ or also $W : U \to V \mid z \curvearrowright z'$.



Lemma 5. *We get $U \parallel V$ for any $i + 1$-cylinder $W : U \to V$. In other words, cylinders form a globular set.*

Note also that the 0-source U and the 0-target V of any $i + 1$-cylinder W are given by $U^\natural = W^\flat$ and $V^\natural = W^\sharp$.

Definition 17. By induction on i, we define the *trivial i-cylinder $\tau\, x : x \curvearrowright x$* for any i-cell x in C:

- if $i = 0$, then $(\tau x)^\natural = 1_x$;
- if $i > 0$, then $(\tau x)^\flat = 1_{x^\flat}$ and $(\tau x)^\sharp = 1_{x^\sharp}$, whereas $[\tau x]$ is the trivial $i - 1$-cylinder $\tau\, [x] : [x] \curvearrowright [x]$.

Lemma 6. *We get $\tau x \parallel \tau y$ for any i-cells $x \parallel y$ in C, and $\tau z : \tau x \to \tau y$ for any $z : x \to y$.*

Definition 18. An i-cylinder is *degenerate* whenever $i = 0$ or $i > 0$ and its source and target are trivial.

Lemma 7 (*Description of Degenerate Cylinders*).

- *For any degenerate i-cylinder $U : x \curvearrowright y$, we get $x \parallel y$ and $U^\natural : x \to y$.*
- *Conversely, any $i+1$-cell $u : x \to y$ yields a unique degenerate i-cylinder $U : x \curvearrowright y$ such that $U^\natural = u$.*

For instance, the unit $1_x : x \to x$ yields the trivial i-cylinder $\tau x : x \curvearrowright x$.

To sum up, we have defined a globular set C^I, whose i-cells are i-cylinders in C, together with homomorphisms $\pi^1, \pi^2 : C^I \to C$ and $\tau : C \to C^I$ such that $\pi^1 \circ \tau = \mathrm{id}_C = \pi^2 \circ \tau$.

$$\begin{tikzcd}
 & C \arrow[dl, "\mathrm{id}_C"'] \arrow[d, "\tau"] \arrow[dr, "\mathrm{id}_C"] & \\
C & C^I \arrow[l, "\pi^1"] \arrow[r, "\pi^2"'] & C
\end{tikzcd}$$

Theorem 3. *There is a structure of ω-category on C^I such that $\pi^1, \pi^2 : C^I \to C$ and $\tau : C \to C^I$ are ω-functors. Moreover, this construction is functorial and π^1, π^2, τ are natural.*

Note that a variant of this construction (*reversible cylinders*) is needed to define the model structure in [15].

The rest of this section is devoted to the proof of this crucial result.

4.2. Concatenation

If $f : C \to D$ is an ω-functor and x is an i-cell in C, we shall write fx for the i-cell $f_i(x)$ in D.

Lemma 8 (*Functoriality*). *Any ω-functor $f : C \to D$ extends to cylinders in a canonical way:*

- *for any i-cylinder $U : x \curvearrowright y$ in C, we get some i-cylinder $f^I U : fx \curvearrowright fy$ in D;*
- *we get $f^I U \parallel f^I V$ whenever $U \parallel V$, and $f^I W : f^I U \to f^I V$ for any $W : U \to V$.*

Moreover, we get $(g \circ f)^I = g^I \circ f^I$ for any $f : C \to D$ and $g : D \to E$, and $\mathrm{id}^I_C = \mathrm{id}_{C^I}$ for any ω-category C. In other words, we get a functor from ω-categories to globular sets and the homomorphisms π^1, π^2 are natural.

In particular, precomposition and postcomposition extend to cylinders. For any 0-cells x, y, z, we get:

- the i-cylinder $u \cdot V$ in $[x, z]$, defined for any 1-cell $u : x \to y$ and for any i-cylinder V in $[y, z]$;
- the i-cylinder $U \cdot v$ in $[x, z]$, defined for any 1-cell $v : y \to z$ and for any i-cylinder U in $[x, y]$.

Those two operations are respectively called *left* and *right action*. By functoriality, we get the following result:

Lemma 9 (*Bimodularity*). *The following identities hold for any 0-cells x, y, z, t:*

- *$(u *_0 v) \cdot W = u \cdot (v \cdot W)$ for any 1-cells $u : x \to y$ and $v : y \to z$, and for any i-cylinder W in $[z, t]$;*
- *$(U \cdot v) \cdot w = U \cdot (v *_0 w)$ for any 1-cells $v : y \to z$ and $w : z \to t$, and for any i-cylinder U in $[x, y]$;*
- *$(u \cdot V) \cdot w = u \cdot (V \cdot w)$ for any 1-cells $u : x \to y$ and $w : z \to t$, and for any i-cylinder V in $[y, z]$.*

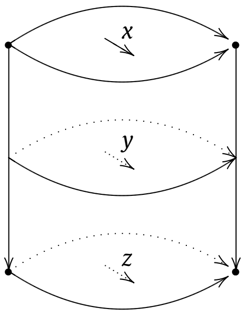
Moreover, we have $1_x \cdot U = U = U \cdot 1_y$ for any 0-cells x, y and for any i-cylinder U in $[x, y]$.

We omit parentheses in such expressions: For instance, $u \cdot v \cdot W$ stands for $u \cdot (v \cdot W)$, and $U \cdot v \cdot w$ for $(U \cdot v) \cdot w$. Moreover, action will always have precedence over other operations: For instance, $u \cdot V * W$ stands for $(u \cdot V) * W$.

Definition 19. By induction on i, we define the i-cylinder $U * V : x \curvearrowright z$, called the *concatenation* of U with V, for any i-cylinders $U : x \curvearrowright y$ and $V : y \curvearrowright z$:

- if $i = 0$, then $(U * V)^\natural = U^\natural *_0 V^\natural$;
- if $i > 0$, then $(U * V)^\flat = U^\flat *_0 V^\flat$ and $(U * V)^\sharp = U^\sharp *_0 V^\sharp$, whereas $[U * V] = [U] \cdot V^\sharp * U^\flat \cdot [V]$.

In both cases, we say that U and V are *consecutive*, and we write $U \triangleright V$.



Lemma 10. *We get $U * U' \parallel V * V'$ for any i-cylinders $U \parallel V$ and $U' \parallel V'$ such that $U \rhd U'$ and $V \rhd V'$, and $W * W' : U * U' \to V * V'$ for any $i+1$-cylinders $W : U \to V$ and $W' : U' \to V'$ such that $W \rhd W'$.*

Lemma 11 (*Compatibility of f^I with $*$ and τ*). *The following identities hold any ω-functor $f : C \to D$:*

- *$f^I(U * V) = f^I U * f^I V$ for any i-cylinders $U \rhd V$ in C;*
- *$f^I(\tau x) = \tau(fx)$ for any i-cell x in C.*

In the cases of precomposition and postcomposition, we get the following result:

Lemma 12 (*Distributivity over $*$ and τ*). *The following identities hold for any 0-cells x, y, z and for any 1-cell $u : x \to y$:*

- *$u \cdot (V * W) = u \cdot V * u \cdot W$ for any i-cylinders $V \rhd W$ in $[y, z]$;*
- *$u \cdot \tau\,[v] = \tau\,[u *_0 v]$ for any $i+1$-cell $v : y \to_0 z$.*

There are similar properties for right action.

Lemma 13 (*Associativity and Units for $*$*).

- *$(U * V) * W = U * (V * W)$ for any i-cylinders $U \rhd V \rhd W$;*
- *$\tau x * U = U = U * \tau y$ for any i-cylinder $U : x \curvearrowright y$.*

Proof. By induction on i. The case $i = 0$ is obvious.
If $i > 0$, the first identity is obtained as follows:

$$\begin{aligned}
[(U * V) * W] &= [U * V] \cdot W^\sharp * (U * V)^\flat \cdot [W] && \text{(definition of } *) \\
&= ([U] \cdot V^\sharp * U^\flat \cdot [V]) \cdot W^\sharp * (U^\flat *_0 V^\flat) \cdot [W] && \text{(definition of } *) \\
&= ([U] \cdot V^\sharp \cdot W^\sharp * U^\flat \cdot [V] \cdot W^\sharp) * U^\flat \cdot V^\flat \cdot [W] && \text{(distributivity over } *) \\
&= [U] \cdot V^\sharp \cdot W^\sharp * (U^\flat \cdot [V] \cdot W^\sharp * U^\flat \cdot V^\flat \cdot [W]) && \text{(induction hypothesis)} \\
&= [U] \cdot (V^\sharp *_0 W^\sharp) * U^\flat \cdot ([V] \cdot W^\sharp * V^\flat \cdot [W]) && \text{(distributivity over } *) \\
&= [U] \cdot (V * W)^\sharp * U^\flat \cdot [V * W] && \text{(definition of } *) \\
&= [U * (V * W)]\,. && \text{(definition of } *)
\end{aligned}$$

The second one is obtained as follows, using distributivity over τ and the induction hypothesis:

$$[\tau x * U] = [\tau x] \cdot U^\sharp * (\tau x)^\flat \cdot [U] = \tau\,[x] \cdot U^\sharp * 1_{x^\flat} \cdot [U] = \tau\left[x *_0 U^\sharp\right] * [U] = [U]\,,$$

and similarly for the third one. $\square$

From now on, we shall omit parentheses in concatenations.

4.3. Compositions and units

Lemma 14. *There are natural isomorphisms $(C \times D)^I \simeq C^I \times D^I$ and $\top^I \simeq \top$, which satisfy the following coherence conditions with the canonical isomorphisms $(C \times D) \times E \simeq C \times (D \times E)$ and $\top \times C \simeq C \simeq C \times \top$:*

$$\begin{tikzcd}
((C \times D) \times E)^I \arrow[r] \arrow[d] & (C \times (D \times E))^I \arrow[d] \\
(C \times D)^I \times E^I \arrow[d] & C^I \times (D \times E)^I \arrow[d] \\
(C^I \times D^I) \times E^I \arrow[r] & C^I \times (D^I \times E^I)
\end{tikzcd}
\qquad
\begin{tikzcd}
(\top \times C)^I \arrow[r] \arrow[d] & C^I \arrow[dd, equal] & (C \times \top)^I \arrow[l] \arrow[d] \\
\top^I \times C^I \arrow[d] & & C^I \times \top^I \arrow[d] \\
\top \times C^I \arrow[r] & C^I & C^I \times \top \arrow[l]
\end{tikzcd}$$

Hence, any ω-bifunctor $f : C \times D \to E$ extends to cylinders in a canonical way. We can apply this to composition: For any 0-cells x, y, z, we get the i-cylinder $U \circledast V$ in $[x, z]$, defined for any i-cylinders U in $[x, y]$ and V in $[y, z]$.

Note also that any 0-cell x in C corresponds to an ω-functor $\iota_x : \top \to C$, from which we get $\iota_x^I : \top \simeq \top^I \to C^I$. It is easy to see that this homomorphism is given by the sequence of trivial i-cylinders $\tau\, 1_x^i$.

In fact, there is also a coherence condition with the symmetry $C \times D \simeq D \times C$, but we shall not use it explicitly.

By functoriality and coherence with the isomorphism $(C \times D) \times E \simeq C \times (D \times E)$, we get the following result:

Lemma 15 (*Associativity of* $\circledast$). *The following identity holds for any 0-cells x, y, z, t, and for any i-cylinders U in $[x, y]$, V in $[y, z]$ and W in $[z, t]$:*

$$(U \circledast V) \circledast W = U \circledast (V \circledast W).$$

Note that $*$ and τ can be defined pairwise in $(C \times D)^I \simeq C^I \times D^I$. By Lemma 11, we get the following result:

Lemma 16 (*Compatibility of* $\circledast$ *with* $*$ *and* τ). *The following identities hold for any 0-cells x, y, z:*

- $(U * U') \circledast (V * V') = (U \circledast V) * (U' \circledast V')$ *for any i-cylinders $U \triangleright U'$ in $[x, y]$ and $V \triangleright V'$ in $[y, z]$;*
- $\tau\,[u] \circledast \tau\,[v] = \tau\,[u *_0 v]$ *for any $i+1$-cells $u : x \to_0 y$ and $v : y \to_0 z$.*

By functoriality and coherence with the isomorphisms $\top \times C \simeq C \simeq C \times \top$, we get the following result:

Lemma 17 (*Representability*). *The following identities hold for any 0-cells x, y, z:*

- $u \cdot V = \tau\, 1^i_{[u]} \circledast V = \tau\left[1_u^{i+1}\right] \circledast V$ *for any 1-cell $u : x \to y$ and for any i-cylinder V in $[y, z]$;*
- $U \cdot v = U \circledast \tau\, 1^i_{[v]} = U \circledast \tau\left[1_v^{i+1}\right]$ *for any 1-cell $v : y \to z$ and for any i-cylinder U in $[x, y]$.*

In other words, the (left and right) action of a 1-cell u is represented by the i-cylinder $\tau\left[1_u^{i+1}\right]$.

For any 0-cells x, y, z, we extend left and right action to higher dimensional cells as follows:

- $u \cdot V = \tau\,[u] \circledast V$ for any $i+1$-cell $u : x \to y$ and for any i-cylinder V in $[y, z]$;
- $U \cdot v = U \circledast \tau\,[v]$ for any $i+1$-cell $v : y \to z$ and for any i-cylinder U in $[x, y]$.

In particular, we get $u \cdot V = \tau\left[1_u^{i+1}\right] \circledast V = 1_u^{i+1} \cdot V$ for any 1-cell $u : x \to y$ and for any i-cylinder V in $[y, z]$, and similarly for the right action. This means that we have indeed extended the action of 1-cells on cylinders.

By associativity of $\circledast$ and compatibility of $\circledast$ with τ, we get the following result:

Lemma 18 (*Extended Bimodularity*). *The first three identities of Lemma 9 extend to higher dimensional cells.*

Lemma 19 (*Extended Distributivity*). *The identities of Lemma 12 extend to higher dimensional cells.*

Proof. The first identity is obtained as follows, using compatibility of $\circledast$ with $*$:

$$u \cdot (V * W) = \tau\,[u] \circledast (V * W) = (\tau\,[u] * \tau\,[u]) \circledast (V * W) = (\tau\,[u] \circledast V) * (\tau\,[u] \circledast W) = u \cdot V * u \cdot W.$$

Similarly, the second one follows from compatibility of $\circledast$ with τ. □

Lemma 20 (*Commutation*). *The following identities hold for any 0-cells x, y, z, for any $i+1$-cells $u, u' : x \to_0 y$ and $v, v' : y \to_0 z$, and for any i-cylinders $U : [u] \curvearrowright [u']$ in $[x, y]$ and $V : [v] \curvearrowright [v']$ in $[y, z]$:*

$$U \cdot v * u' \cdot V = U \circledast V = u \cdot V * U \cdot v'.$$

Proof. The first identity is obtained as follows, using compatibility of $\circledast$ with $*$:

$$U \cdot v * u' \cdot V = (U \circledast \tau\,[v]) * \left(\tau\left[u'\right] \circledast V\right) = \left(U * \tau\left[u'\right]\right) \circledast (\tau\,[v] * V) = U \circledast V,$$

and similarly for the second one. □

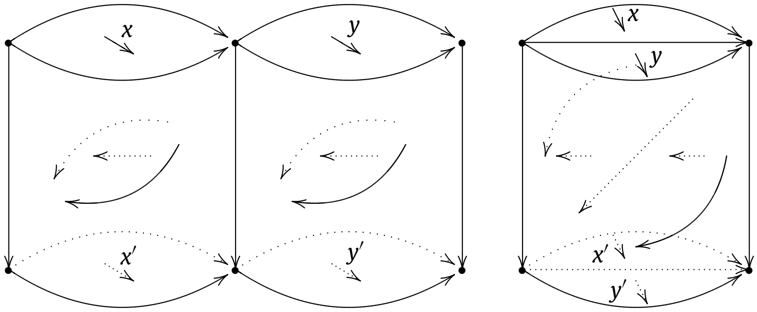
From now on, we assume that $j > i$.

Definition 20. By induction on i, we define the j-cylinder $U *_i V : R \to_i T \mid x *_i y \curvearrowright x' *_i y'$ for any j-cylinders $U : R \to_i S \mid x \curvearrowright x'$ and $V : S \to_i T \mid y \curvearrowright y'$:

- $(U *_0 V)^\flat = U^\flat = R^\natural$ and $(U *_0 V)^\sharp = V^\sharp = T^\natural$, whereas $[U *_0 V] = x \cdot [V] * [U] \cdot y'$;
- if $i > 0$, then $(U *_i V)^\flat = U^\flat = V^\flat$ and $(U *_i V)^\sharp = U^\sharp = V^\sharp$, whereas $[U *_i V] = [U] *_{i-1} [V]$.

In both cases, we say that U and V are *i-composable*, and we write $U \triangleright_i V$.

The following picture shows the 0-composition and 1-composition of 2-cylinders:



Lemma 21. *We get $U *_i U' \parallel V *_i V'$ for any j-cylinders $U \parallel V$ and $U' \parallel V'$ such that $U \rhd_i U'$ (so that $V \rhd_i V'$), and $W *_i W' : U *_i U' \to V *_i V'$ for any $j+1$-cylinders $W : U \to V$ and $W' : U' \to V'$.*

Definition 21. By induction on *i*, we define the $i+1$-cylinder $1_U : U \to U \mid 1_x \curvearrowright 1_y$ for any *i*-cylinder $U : x \curvearrowright y$:

- if $i = 0$, then $(1_U)^\flat = (1_U)^\sharp = U^\natural$, whereas $[1_U] = \tau\left[U^\natural\right]$;
- if $i > 0$, then $(1_U)^\flat = U^\flat$ and $(1_U)^\sharp = U^\sharp$, whereas $[1_U] = 1_{[U]}$.

We write 1_U^{i+1} for 1_U, and we inductively define $1_U^{j+1} : 1_U^j \to 1_U^j \mid 1_x^{j+1} \curvearrowright 1_y^{j+1}$ by $1_U^{j+1} = 1_{1_U^j}$ for all $j > i$.

Lemma 22 (*Compatibility of τ with $*_i$ and Units*).

- *$\tau(u *_i v) = \tau u *_i \tau v$ for any j-cells $u \rhd_i v$;*
- *$\tau 1_x = 1_{\tau x}$ for any i-cell x.*

Proof. By induction on *i*.

If $i = 0$, the first identity is obtained as follows, using distributivity over τ:

$$[\tau(u *_0 v)] = \tau\,[u *_0 v] = \tau\,[u *_0 v] * \tau\,[u *_0 v] = u \cdot \tau\,[v] * \tau\,[u] \cdot v = u \cdot [\tau v] * [\tau u] \cdot v = [\tau u *_0 \tau v]\,.$$

The second one is obtained as follows: $[\tau 1_x] = \tau\,[1_x] = \tau\left[(\tau x)^\natural\right] = [1_{\tau x}]$.

If $i > 0$, we apply the induction hypothesis. □

Now, we write 1_x^i for *x* whenever *x* is an *i*-cell, so that the following result holds for $j = i + 1$:

Lemma 23. *For all $j > i$ and for any i-cylinder $U : x \curvearrowright y$, we get the following characterization of 1_U^j:*

- *if $i = 0$, then $(1_U^j)^\flat = (1_U^j)^\sharp = U^\natural$, whereas $\left[1_U^j\right] = \tau\left[1_{U^\natural}^j\right]$;*
- *if $i > 0$, then $(1_U^j)^\flat = U^\flat$ and $(1_U^j)^\sharp = U^\sharp$, whereas $\left[1_U^j\right] = 1_{[U]}^{j-1}$.*

This is easily proved by induction on *j*, using compatibility of τ with units.

Lemma 24 (*Associativity and Units for $*_i$*).

- *$(U *_i V) *_i W = U *_i (V *_i W)$ for any j-cylinders $U \rhd_i V \rhd_i W$;*
- *$1_U^j *_i W = W = W *_i 1_V^j$ for any j-cylinder $W : U \to_i V$.*

Proof. By induction on *i*.

If $i = 0$, the first identity is obtained as follows (with $U : x \curvearrowright x'$, $V : y \curvearrowright y'$ and $W : z \curvearrowright z'$):

$$\begin{aligned}
[(U *_0 V) *_0 W] &= (x *_0 y) \cdot [W] * [U *_0 V] \cdot z' && \text{(definition of } *_0\text{)}\\
&= x \cdot y \cdot [W] * (x \cdot [V] * [U] \cdot y') \cdot z' && \text{(definition of } *_0\text{)}\\
&= x \cdot y \cdot [W] * x \cdot [V] \cdot z' * [U] \cdot y' \cdot z' && \text{(distributivity over } *\text{)}\\
&= x \cdot (y \cdot [W] * [V] \cdot z') * [U] \cdot y' \cdot z' && \text{(distributivity over } *\text{)}\\
&= x \cdot [V * W] * [U] \cdot (y' *_0 z') && \text{(definition of } *_0\text{)}\\
&= [U *_0 (V *_0 W)]\,. && \text{(definition of } *_0\text{)}
\end{aligned}$$

The second one is obtained as follows (with $W : x \curvearrowright y$ and $U : x^\flat \curvearrowright y^\flat$), using distributivity over τ:

$$\left[1_U^j *_0 W\right] = 1_{x^\flat}^j \cdot [W] * \left[1_U^j\right] \cdot y = 1_{x^\flat} \cdot [W] * \tau\left[1_{U^\natural}^j\right] \cdot y = [W] * \tau\left[1_{U^\natural}^j *_0 y\right] = [W]\,,$$

and similarly for the third one.

If $i > 0$, we apply the induction hypothesis. □

4.4. Interchange

Lemma 25 (*Compatibility of f^I with $*_i$ and Units*). *The following identities hold any ω-functor $f : C \to D$:*

- $f^I(U *_i V) = f^I U *_i f^I V$ *for any j-cylinders $U \rhd_i V$ in C;*
- $f^I 1_U = 1_{f^I U}$ *for any i-cylinder U in C.*

In the cases of precomposition and postcomposition, we get the following result:

Lemma 26 (*Distributivity over $*_i$ and Units*). *The following identities hold for any 0-cells x, y, z and for any 1-cell $u : x \to y$:*

- $u \cdot (V *_i W) = u \cdot V *_i u \cdot W$ *for any j-cylinders $V \rhd_i W$ in $[y, z]$;*
- $u \cdot 1_V = 1_{u \cdot V}$ *for any i-cylinder V in $[y, z]$.*

There are similar properties for right action.

Lemma 27 (*Compatibility of $*$ with $*_i$ and Units*).

- $(U *_i V) * (U' *_i V') = (U * U') *_i (V * V')$ *for any j-cylinders $U \rhd_i V$ and $U' \rhd_i V'$ such that $U \rhd U'$ and $V \rhd V'$;*
- $1_U * 1_V = 1_{U*V}$ *for any i-cylinders $U \rhd V$.*

Proof. By induction on i.

If $i = 0$, the first identity is obtained as follows (with $U : x \curvearrowright x'$, $U' : x' \curvearrowright x''$, $V : y \curvearrowright y'$ and $V' : y' \curvearrowright y''$):

$$\begin{aligned}
\left[(U *_0 V) * (U' *_0 V')\right] &= [U *_0 V] \cdot (U' *_0 V')^\sharp * (U *_0 V)^\flat \cdot \left[U' *_0 V'\right] && \text{(definition of } *) \\
&= (x \cdot [V] * [U] \cdot y') \cdot V'^\sharp * U^\flat \cdot \left(x' \cdot \left[V'\right] * \left[U'\right] \cdot y''\right) && \text{(definition of } *_0) \\
&= x \cdot [V] \cdot V'^\sharp * [U] \cdot y' \cdot V'^\sharp * U^\flat \cdot x' \cdot \left[V'\right] * U^\flat \cdot \left[U'\right] \cdot y'' && \text{(distributivity over } *) \\
&= x \cdot [V] \cdot V'^\sharp * x \cdot V^\flat \cdot \left[V'\right] * [U] \cdot U'^\sharp \cdot y'' * U^\flat \cdot \left[U'\right] \cdot y'' && \text{(commutation)} \\
&= x \cdot \left([V] \cdot V'^\sharp * V^\flat \cdot \left[V'\right]\right) * \left([U] \cdot U'^\sharp * U^\flat \cdot \left[U'\right]\right) \cdot y'' && \text{(distributivity over } *) \\
&= x \cdot \left[V * V'\right] * \left[U * U'\right] \cdot y'' && \text{(definition of } *) \\
&= \left[(U * U') *_0 (V * V')\right]. && \text{(definition of } *_0)
\end{aligned}$$

In the commutation step, we use the fact that $U^\sharp = V^\flat$ and $U'^\sharp = V'^\flat$ since $U \rhd_0 V$ and $U' \rhd_0 V'$.

The second one is obtained as follows, using distributivity over τ:

$$\begin{aligned}
[1_U * 1_V] &= [1_U] \cdot (1_V)^\sharp * (1_U)^\flat \cdot [1_V] = \tau \left[U^\natural\right] \cdot V^\natural * U^\natural \cdot \tau \left[V^\natural\right] \\
&= \tau \left[U^\natural *_0 V^\natural\right] * \tau \left[U^\natural *_0 V^\natural\right] = \tau \left[U^\natural *_0 V^\natural\right] = \tau \left[(U * V)^\natural\right] = [1_{U*V}] .
\end{aligned}$$

If $i > 0$, the first identity is obtained as follows:

$$\begin{aligned}
\left[(U *_i V) * (U' *_i V')\right] &= [U *_i V] \cdot (U' *_i V')^\sharp * (U *_i V)^\flat \cdot \left[U' *_i V'\right] && \text{(definition of } *) \\
&= ([U] *_{i-1} [V]) \cdot U'^\sharp * U^\flat \cdot \left(\left[U'\right] *_{i-1} \left[V'\right]\right) && \text{(definition of } *_i) \\
&= ([U] \cdot U'^\sharp *_{i-1} [V] \cdot U'^\sharp) * \left(U^\flat \cdot \left[U'\right] *_{i-1} U^\flat \cdot \left[V'\right]\right) && \text{(distributivity over } *_{i-1}) \\
&= \left([U] \cdot U'^\sharp * U^\flat \cdot \left[U'\right]\right) *_{i-1} \left([V] \cdot U'^\sharp * U^\flat \cdot \left[V'\right]\right) && \text{(induction hypothesis)} \\
&= \left[U * U'\right] *_{i-1} \left[V * V'\right] && \text{(definition of } *) \\
&= \left[(U * U') *_i (V * V')\right]. && \text{(definition of } *_i)
\end{aligned}$$

In the penultimate step, we use the fact that $U^\flat = V^\flat$ and $U'^\sharp = V'^\sharp$ since $U \rhd_i V$ and $U' \rhd_i V'$.

The second one is obtained as follows, using distributivity over units and the induction hypothesis:

$$\begin{aligned}
[1_U * 1_V] &= [1_U] \cdot (1_V)^\sharp * (1_U)^\flat \cdot [1_V] = 1_{[U]} \cdot V^\sharp * U^\flat \cdot 1_{[V]} \\
&= 1_{[U] \cdot V^\sharp} * 1_{U^\flat \cdot [V]} = 1_{[U] \cdot V^\sharp * U^\flat \cdot [V]} = 1_{[U*V]} = [1_{U*V}] . \quad \square
\end{aligned}$$

Note that $*_i$ and units can be defined pairwise in $(C \times D)^I \simeq C^I \times D^I$. By Lemma 25, we get the following result:

Lemma 28 (*Compatibility of $\circledast$ with $*_i$ and Units*). *The following identities hold for any 0-cells x, y, z:*

- $(U *_i U') \circledast (V *_i V') = (U \circledast V) *_i (U' \circledast V')$ *for any j-cylinders $U \rhd_i U'$ in $[x, y]$ and $V \rhd_i V'$ in $[y, z]$;*
- $1_U \circledast 1_V = 1_{U \circledast V}$ *for any i-cylinders U in $[x, y]$ and V in $[y, z]$.*

Lemma 29 (*Compatibility of $\cdot$ with $*_i$ and Units*). *The following identities hold for any 0-cells x, y, z:*

- *$(u *_{i+1} u') \cdot (V *_i V') = u \cdot V *_i u' \cdot V'$ for any $j+1$-cells $u, u' : x \to_0 y$ such that $u \rhd_{i+1} u'$ and for any j-cylinders $V \rhd_i V'$ in $[y, z]$;*
- *$1_u \cdot 1_V = 1_{u \cdot V}$ for any $i+1$-cell $u : x \to_0 y$ and for any i-cylinder V in $[y, z]$.*

There are similar properties for right action.

Proof. The first identity is obtained as follows:

$$\begin{aligned}
(u *_{i+1} u') \cdot (V *_i V') &= \tau \left[u *_{i+1} u'\right] \circledast (V *_i V') && \text{(definition of } \cdot) \\
&= \tau \left([u] *_i \left[u'\right]\right) \circledast (V *_i V') && \text{(definition of } *_i \text{ in } [x, y]) \\
&= \left(\tau\, [u] *_i \tau \left[u'\right]\right) \circledast (V *_i V') && \text{(compatibility of } \tau \text{ with } *_i) \\
&= (\tau\, [u] \circledast V) *_i \left(\tau \left[u'\right] \circledast V'\right) && \text{(compatibility of } \circledast \text{ with } *_i) \\
&= u \cdot V *_i u' \cdot V'. && \text{(definition of } \cdot)
\end{aligned}$$

The second one is obtained as follows, using compatibility of τ and $\circledast$ with units:

$$1_u \cdot 1_V = \tau\, [1_u] \circledast 1_V = \tau\, 1_{[u]} \circledast 1_V = 1_{\tau[u]} \circledast 1_V = 1_{\tau[u] \circledast V} = 1_{u \cdot V}. \quad \square$$

Now we assume that $k > j > i$.

Lemma 30 (*Interchange Laws*).

- *$(U *_j U') *_i (V *_j V') = (U *_i V) *_j (U' *_i V')$ for any k-cylinders $U \rhd_j U'$ and $V \rhd_j V'$ such that $U \rhd_i V$;*
- *$1_U *_i 1_V = 1_{U *_i V}$ for any j-cylinders $U \rhd_i V$.*

Proof. By induction on i.

If $i = 0$, the first identity is obtained as follows (with $U : x \curvearrowright y$, $U' : x' \curvearrowright y'$, $V : z \curvearrowright t$ and $V' : z' \curvearrowright t'$):

$$\begin{aligned}
\left[(U *_j U') *_0 (V *_j V')\right] &= (x *_j x') \cdot \left[V *_j V'\right] * \left[U *_j U'\right] \cdot (t *_j t') && \text{(definition of } *_0) \\
&= (x *_j x') \cdot \left([V] *_{j-1} \left[V'\right]\right) * \left([U] *_{j-1} \left[U'\right]\right) \cdot (t *_j t') && \text{(definition of } *_j) \\
&= \left(x \cdot [V] *_{j-1} x' \cdot \left[V'\right]\right) * \left([U] \cdot t *_{j-1} \left[U'\right] \cdot t'\right) && \text{(compatibility of } \cdot \text{ with } *_{j-1}) \\
&= (x \cdot [V] * [U] \cdot t) *_{j-1} \left(x' \cdot \left[V'\right] * \left[U'\right] \cdot t'\right) && \text{(compatibility of } * \text{ with } *_{j-1}) \\
&= [U *_0 V] *_{j-1} \left[U' *_0 V'\right] && \text{(definition of } *_0) \\
&= \left[(U *_0 V) *_j (U' *_0 V')\right]. && \text{(definition of } *_j)
\end{aligned}$$

The second one is obtained as follows (with $U : x \curvearrowright x'$ and $V : y \curvearrowright y'$), using compatibility of $\cdot$ and $*$ with units:

$$\begin{aligned}
[1_U *_0 1_V] &= 1_x \cdot [1_V] * [1_U] \cdot 1_{y'} = 1_x \cdot 1_{[V]} * 1_{[U]} \cdot 1_{y'} \\
&= 1_{x \cdot [V]} * 1_{[U] \cdot y'} = 1_{x \cdot [V] * [U] \cdot y'} = 1_{[U *_0 V]} = \left[1_{U *_0 V}\right].
\end{aligned}$$

If $i > 0$, we apply the induction hypothesis. $\square$

To sum up, we have the following results:

- C^I is an ω-category by Lemmas 24 and 30;
- π^1, π^2 are ω-functors by construction and τ by Lemma 22;
- C^I is functorial by Lemmas 8 and 25;
- π^1, π^2 are natural by Lemma 8 and τ by Lemma 11.

Hence, we have proved Theorem 3.

5. Homotopy

Definition 22. Let $f, g : E \to C$ be two ω-functors. A *(directed) homotopy* from f to g is an ω-functor $h : E \to C^I$ such that $\pi^1_C \circ h = f$ and $\pi^2_C \circ h = g$. The existence of such a homotopy is denoted by $f \rightsquigarrow g$.

In other words, $f \rightsquigarrow g$ if and only if there is an $h : E \to C^I$ such that the following diagram commutes, with $\pi = \left(\pi^1, \pi^2\right)$:

$$\begin{tikzcd}
 & C^I \arrow[d, "\pi"] \\
E \arrow[ur, dashed, "h"] \arrow[r, "(f,g)"'] & C^2
\end{tikzcd}$$

We first turn to the proof of Proposition 2, Section 2.3 and Proposition 9, Section 3.4, a generalization of the former. In both cases we are given a polygraph S and ω-functors $p : C \to D, f, g : S^* \to C$, where p satisfies some lifting properties, and we need to build an $h : S^* \to C^I$ making the following diagram commutative:

$$\begin{tikzcd} & C^I \arrow[d, "\pi"] \\ S^* \arrow[ur, dashed, "h"] \arrow[r, "{(f,g)}"'] & C^2 \end{tikzcd}$$

Now π is not an acyclic fibration in general: therefore, we need a new ω-functor $\pi_{/p}$, restricting π to some ω-categories depending on p and having the desired lifting properties.

5.1. Restriction of the projection

In this section, we define the abovementioned ω-functor $\pi_{/p}$ and establish its lifting properties.

Thus, let $p : C \to D$ by any ω-functor, and $\Delta : D \to D^2$ the diagonal map: $x \mapsto (x, x)$. We define a new ω-category $C^2_{/p}$ together with ω-functors a and $\Delta^* p^2$ by the following pullback square:

$$\begin{tikzcd} C^2_{/p} \arrow[r, "a"] \arrow[d, "\Delta^* p^2"'] & C^2 \arrow[d, "p^2"] \\ D \arrow[r, "\Delta"'] & D^2 \end{tikzcd} \tag{3}$$

Concretely, an i-cell of $C^2_{/p}$ amounts to a pair (x, y) of i-cells in C such that $p(x) = p(y)$. Likewise, we define $C^I_{/p}$, b and $\tau^* p^I$ by the following pullback:

$$\begin{tikzcd} C^I_{/p} \arrow[r, "b"] \arrow[d, "\tau^* p^I"'] & C^I \arrow[d, "p^I"] \\ D \arrow[r, "\tau"'] & D^I \end{tikzcd} \tag{4}$$

Here an i-cell of $C^I_{/p}$ amounts to an i-cylinder U of C such that $p^I(U)$ is a trivial i-cylinder of D.

Lemma 31. *There is a unique ω-functor $\pi_{/p} : C^I_{/p} \to C^2_{/p}$ such that the following cube commutes:*

$$\begin{tikzcd} C^I_{/p} \arrow[rr, "b"] \arrow[dr, "\pi_{/p}"] \arrow[dd, "\tau^* p^I"'] & & C^I \arrow[dr, "\pi"] \arrow[dd, dotted, "p^I" near start] & \\ & C^2_{/p} \arrow[rr, "a" near start] \arrow[dd, "\Delta^* p^2" near start] & & C^2 \arrow[dd, "p^2"] \\ D \arrow[rr, dotted, "\tau" near start] \arrow[dr, "1_D"'] & & D^I \arrow[dr, dotted, "\pi"] & \\ & D \arrow[rr, "\Delta"'] & & D^2 \end{tikzcd} \tag{5}$$

Proof. The front and back squares are respectively (3) and (4), hence commute, by definition. The right-hand square commutes because π is a natural transformation, and the bottom square commutes because $\pi \circ \tau = \Delta$. Therefore

$$p^2 \circ \pi \circ b = \Delta \circ \tau^* p^I$$

and because (3) is a pullback, we get the required connecting morphism $\pi_{/p}$. $\square$

To sum up, we have associated to each p a unique ω-functor $\pi_{/p}$ making (5) commutative. Precisely, if $U : x \curvearrowright y$ is an i-cell of $C^I_{/p}$, then (x, y) is an i-cell in $C^2_{/p}$ and $\pi_{/p}(U) = (x, y)$.

The following result shows how lifting properties of p transfer to $\pi_{/p}$. Geometrically speaking, Lemma 32 says that certain "boxes" consisting of two parallel i-cylinders, with top and bottom $i + 1$-cells, may be filled by an $i + 1$ cylinder.

For any $p : C \to D$ and any 0-cells z, z' of C, we denote by $p_{z,z'}$ the ω-functor from $[z, z']$ to $[p(z), p(z')]$ induced by p.

Lemma 32. *Let $p : C \to D$ and suppose that*

- *$U : x \curvearrowright y$, $U' : x' \curvearrowright y'$ are i-cylinders defining parallel i-cells of $C^I_{/p}$;*
- *$(u, v) : (x, y) \to \left(x', y'\right)$ is an $i + 1$-cell of $C^2_{/p}$;*
- *$p_{u^\flat, v^\sharp}$ has the lifting property.*

Then, we get an $i + 1$-cylinder $W : U \to U' \mid u \curvearrowright v$ defining an $i + 1$-cell in $C^I_{/p}$.

Proof. We proceed by induction on i.

- Suppose that $i = 0$. In that case U, U' are 0-cells of $C^I_{/p}$, and $u : x \to x'$, $v : y \to y'$ 1-cells of C such that $p(u) = p(v)$. Thus $u_1 = u *_0 U'^\natural$, $v_1 = U^\natural *_0 v$ are parallel 1-cells of C with $u_1, v_1 : u^\flat \to v^\sharp$. As U, U' belong to $C^I_{/p}$, $p(U^\natural)$ and $p(U'^\natural)$ are identities, so that $p(u_1) = p(u) = p(v) = p(v_1)$. Thus $p_{u^\flat, v^\sharp}[u_1] = p_{u^\flat, v^\sharp}[v_1]$ and because $p_{u^\flat, v^\sharp}$ has the lifting property, we get a 1-cell $[w] : [u_1] \to [v_1]$ of $[u^\flat, v^\sharp]$ such that $p_{u^\flat, v^\sharp}[w] = 1_{p[u_1]}$.

  Hence, there is a 1-cylinder $W : U \to U' \mid u \curvearrowright v$ given by $W^\flat = U^\natural$, $W^\sharp = U'^\natural$, $W^\natural = w$ and defining a 1-cell in $C^I_{/p}$.

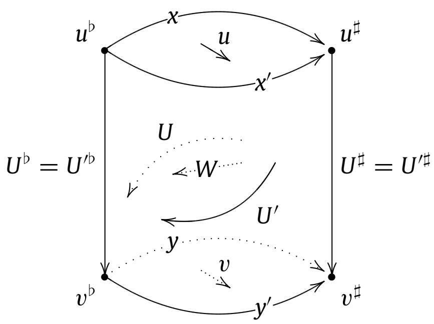
$$\begin{tikzcd} x \arrow[r, "u"] \arrow[d, "U^\natural"'] \arrow[dr, phantom, "\swarrow\, W^\natural = w"] & x' \arrow[d, "U'^\natural"] \\ y \arrow[r, "v"'] & y' \end{tikzcd}$$

- Suppose that $i > 0$ and that the property holds in dimension $i - 1$. Consider now the ω-categories $E = [u^\flat, v^\sharp]$, $F = [p(u^\flat), p(v^\sharp)]$, and let $q = p_{u^\flat, v^\sharp} : E \to F$. By definition, we get two $i - 1$-cylinders in E:

$$[U] : [x] \cdot U^\sharp \curvearrowright U^\flat \cdot [y]\,,$$
$$\left[U'\right] : \left[x'\right] \cdot U'^\sharp \curvearrowright U'^\flat \cdot \left[y'\right].$$

  Now $[U] \parallel \left[U'\right]$; also $q^I([U])$ and $q^I\left(\left[U'\right]\right)$ are trivial cylinders of F, so that $[U]$, $\left[U'\right]$ define parallel $i - 1$-cells of $E^I_{/q}$. Moreover $\left([u] \cdot U^\sharp, U^\flat \cdot [v]\right) : \left([x] \cdot U^\sharp, U^\flat \cdot [y]\right) \to \left(\left[x'\right] \cdot U^\sharp, U^\flat \cdot \left[y'\right]\right)$ is an i-cell of $E^2_{/q}$.

  As q has the lifting property, so does $q_{z,z'}$ for any 0-cells z, z'. Therefore, the induction hypothesis applies to q and we get an i-cylinder $V : [U] \to \left[U'\right] \mid [x] \cdot U^\sharp \curvearrowright U^\flat \cdot \left[y'\right]$ defining an i-cell of of $E^I_{/q}$; whence an $i + 1$-cylinder $W : U \to U' \mid u \curvearrowright v$ given by $W^\flat = U^\flat = U'^\flat$, $W^\sharp = U^\sharp = U'^\sharp$ and $[W] = V$. By construction, W defines an $i + 1$-cell in $C^I_{/p}$.



$\square$

5.2. Acyclic case

We may now prove Proposition 2:

For any $p : C \to D$ and $f, g : S^* \to C$ such that $p \circ f = p \circ g$ and p has the lifting property, we get a *homotopy* $f \rightsquigarrow g$.

The crucial point is the following result:

Lemma 33. *If $p : C \to D$ has the lifting property, then $\pi_{/p}$ is an acyclic fibration.*

Proof. Suppose that $p : C \to D$ has the lifting property.

- Let z be a 0-cell in $C^2_{/p}$: it is a pair $z = (x, y)$ of 0-cells in C such that $p(x) = p(y)$. As p has the lifting property, there is a 1-cell $u : x \to y$ such that $p(u) = 1_{p(x)} = 1_{p(y)}$, hence a 0-cylinder U in C^I such that $U^\natural = u$ and $p^I(U) = \tau(p(x))$. Therefore U is a 0-cell of $C^I_{/p}$ such that $(\pi_{/p})_0(U) = z$, and $(\pi_{/p})_0$ is onto.
- The fact that $\pi_{/p}$ has the lifting property is an immediate consequence of Lemma 32. □

Consider now $p : C \to D$ and $f, g : S^* \to C$ such that $p \circ f = p \circ g = k$. In other words, the following diagram commutes:

$$\begin{tikzcd} S^* \arrow[r, "{(f,g)}"] \arrow[d, "k"'] & C^2 \arrow[d, "p^2"] \\ D \arrow[r, "\Delta"'] & D^2 \end{tikzcd}$$

Hence the pullback square (3) yields a unique ω-functor $l : S^* \to C^2_{/p}$ such that $(f, g) = a \circ l$. If p has the lifting property, then $\pi_{/p}$ is an acyclic fibration by Lemma 33, and Proposition 1, Section 2.3 yields an ω-functor $\hat{l}$ such that $l = \pi_{/p} \circ \hat{l}$. Thus, we get a commutative diagram:

$$\begin{tikzcd} & C^I_{/p} \arrow[r, "b"] \arrow[d, "\pi_{/p}"] & C^I \arrow[d, "\pi"] \\ S^* \arrow[ru, "\hat{l}"] \arrow[r, "l"] \arrow[rr, bend right, "{(f,g)}"'] & C^2_{/p} \arrow[r, "a"] & C^2 \end{tikzcd}$$

By defining $h = b \circ \hat{l} : S^* \to C^I$, we get $\pi \circ h = (f, g)$. Hence $f \rightsquigarrow g$ as expected.

5.3. *Relative case*

We now adapt the above arguments to the more general situation of Proposition 9:

> For any $p : C \to D$ and $f, g : S^* \to C$ such that $p \circ f = p \circ g$ and p satisfies the lifting property with respect to $\big(f_0(S^*_0), g_0(S^*_0)\big)$, we get a homotopy $f \rightsquigarrow g$.

We first state a generalized version of Lemma 33:

Lemma 34. *Let $p : C \to D$, $\mathcal{X}, \mathcal{Y} \subset C_0$, $\mathcal{Z} = a_0^{-1}(\mathcal{X} \times \mathcal{Y}) \subset (C^2_{/p})_0$ and $\mathcal{U} = (\pi_{/p})_0^{-1}(\mathcal{Z}) \subset (C^I_{/p})_0$. If p has the lifting property with respect to $(\mathcal{X}, \mathcal{Y})$, then*

1. *$\mathcal{Z} \subset (\pi_{/p})_0(\mathcal{U})$;*
2. *$\pi_{/p}$ has the lifting property with respect to $\mathcal{U}$.*

Proof. Suppose that p has the lifting property with respect to $(\mathcal{X}, \mathcal{Y})$.

- Consider a 0-cell z in $\mathcal{Z} = a_0^{-1}(\mathcal{X} \times \mathcal{Y})$. It is a pair $z = (x, y)$ of 0-cells in C such that $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $p(x) = p(y)$. As p has the lifting property with respect to $(\mathcal{X}, \mathcal{Y})$, there is a 1-cell $u : x \to y$ such that $p(u) = 1_{p(x)} = 1_{p(y)}$, hence a 0-cylinder U such that $U^\natural = u$ and $p^I(U) = \tau(p(x))$. Therefore U is a 0-cell of $C^I_{/p}$ and $(\pi_{/p})_0(U) = z$, so that $U \in \mathcal{U}$ and $z \in (\pi_{/p})_0(\mathcal{U})$. This proves the first point.
- The second part follows immediately from Lemma 32. □

If $f, g : S^* \to C$ satisfy $p \circ f = p \circ g$, we get as above a factorization $(f, g) = a \circ l$ where $l : S^* \to C^2_{/p}$.

Suppose now that p has the lifting property with respect to $(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X} = f_0(S^*_0)$ and $\mathcal{Y} = g_0(S^*_0)$. Define $\mathcal{Z}$ and $\mathcal{U}$ as in Lemma 34: $l_0(S^*_0) \subset \mathcal{Z}$ by construction of l. By Lemma 34, Proposition 8 applies and we get an ω-functor $\hat{l}$ such that $l = \pi_{/p} \circ \hat{l}$.

By defining $h = b \circ \hat{l}$ we get as above the desired homotopy from f to g.

5.4. Chain-homotopy

In this section, we prove Proposition 3, Section 2.4:

> for any polygraphs S, T, and ω-functors $f, g : S^* \to T^*$ such that $f \rightsquigarrow g$, the $\mathbb{Z}$-linear maps $f^{\mathrm{ab}}, g^{\mathrm{ab}} : \mathbb{Z}S \to \mathbb{Z}T$ are chain-homotopic.

We first need a few additional results about abelianization and cylinders. Consider the truncation endofunctor $\mathbf{T}$ of the category of ω-categories, defined by $(\mathbf{T}C)_i = C_{i+1}$ for each ω-category C and $i \geq 0$. For any 0-cells x, y of C, $[x, y]$ is a full subcategory of $\mathbf{T}C$. On the other hand, if S is a polygraph, there are linearization maps:

$$\lceil . \rfloor : S_i^* \to \mathbb{Z}S_i \tag{6}$$

in each dimension i (see Section 2.4). Now $\mathbf{T}C$ is *not* in general a free ω-category, even if $C = S^*$; however, we may extend the linearization process to all ω-categories of the form $\mathbf{T}^k C$, for $k \geq 0$, by considering any i-cell x of $\mathbf{T}^k C$, as an $i+k$-cell of C. Hence, whenever $C = S^*$, we get from (6) linearization maps

$$\lceil . \rfloor : (\mathbf{T}^k C)_i \to \mathbb{Z}S_{i+k}. \tag{7}$$

Note that these maps still take compositions in $\mathbf{T}^k C$ to sums.

Lemma 35. *Let $C = S^*$ be a free ω-category, and $k \geq 0$ an integer. If $i > 0$ and $W : U \to V \mid x \curvearrowright y$ is an i-cylinder of $\mathbf{T}^k C$, then:*

$$\lceil \sigma_i W^\natural \rfloor = \lceil x \rfloor + \lceil V^\natural \rfloor, \tag{8}$$

$$\lceil \tau_i W^\natural \rfloor = \lceil U^\natural \rfloor + \lceil y \rfloor. \tag{9}$$

Proof. We proceed by induction on $i \geq 1$.

- Suppose that $i = 1$, and let $W : U \to V \mid x \curvearrowright y$ be a 1-cylinder of $\mathbf{T}^k C$. Thus $W^\natural$ is a 2-cell of $\mathbf{T}^k C$, and

  $$W^\natural : x *_0 V^\natural \to U^\natural *_0 y.$$

  Hence $\sigma_1 W^\natural = x *_0 V^\natural$ and $\tau_1 W^\natural = U^\natural *_0 y$, which, by linearization, gives (8) and (9).
- Suppose that $i > 1$, and that (8) and (9) hold for $i-1$. Let $W : U \to V \mid x \curvearrowright y$ be an i-cylinder of $\mathbf{T}^k C$. We get an $i-1$-cylinder $[W] : [U] \to [V] \mid [x] \cdot V^\sharp \curvearrowright U^\flat \cdot [y]$ of $[x^\flat, y^\sharp]$. We may see $[W]$ as an $i-1$-cylinder of $\mathbf{T}^{k+1} C$, so that the induction hypothesis applies and we get

  $$\lceil \sigma_{i-1} [W]^\natural \rfloor = \lceil [x] \cdot V^\sharp \rfloor + \lceil [V]^\natural \rfloor$$
  $$\lceil \tau_{i-1} [W]^\natural \rfloor = \lceil [U]^\natural \rfloor + \lceil U^\flat \cdot [y] \rfloor.$$

  Now $\sigma_{i-1} [W]^\natural$, $[x] \cdot V^\sharp$ and $[V]^\natural$ are $i-1$-cells of $\mathbf{T}^{k+1} C$, which can be seen as i-cells in $\mathbf{T}^k C$, respectively $\sigma_i W^\natural$, $x *_0 V^\sharp$ and $V^\natural$. As $i > 1$, $V^\sharp$ is a unit. Therefore $\lceil V^\sharp \rfloor = 0$, and

  $$\begin{aligned} \lceil \sigma_i W^\natural \rfloor &= \lceil x *_0 V^\sharp \rfloor + \lceil V^\natural \rfloor, \\ &= \lceil x \rfloor + \lceil V^\sharp \rfloor + \lceil V^\natural \rfloor, \\ &= \lceil x \rfloor + \lceil V^\natural \rfloor. \end{aligned}$$

  Thus, we get (8), and the same argument applies to (9). $\square$

Lemma 36. *If U, V are j-composable cylinders in a free ω-category, then $\lceil (U *_j V)^\natural \rfloor = \lceil U^\natural \rfloor + \lceil V^\natural \rfloor$.*

Proof. One first checks that the corresponding relation holds for concatenation, namely

$$\lceil (U * V)^\natural \rfloor = \lceil U^\natural \rfloor + \lceil V^\natural \rfloor.$$

This proves the case $j = 0$, after Definition 20, and the general case follows by induction on j. $\square$

Lemma 37. *If S, T are polygraphs and $h : S^* \to (T^*)^I$ is an ω-functor, then for each $i \geq 0$, there is a $\mathbb{Z}$-linear map $\theta_i : \mathbb{Z}S_i \to \mathbb{Z}T_{i+1}$ satisfying*

$$\theta_i \lceil x \rfloor = \lceil h_i(x)^\natural \rfloor \tag{10}$$

whenever x is an i-cell of S^*.

Proof. There is a unique $\theta_i : \mathbb{Z}S_i \to \mathbb{Z}T_{i+1}$ such that $\theta_i \lceil \xi \rfloor = \lceil h_i(\xi)^\natural \rfloor$ for each $\xi \in S_i$. Let us show (10) by structural induction on $x \in S_i^*$:

- if x is an i-generator, (10) holds by definition;

- if x is a unit, then so is $h_i(x)$, because h is an ω-functor: therefore $h_i(x)^\natural$ is a unit in T^*_{i+1}, so that both sides of (10) vanish;
- if x decomposes as $y *_j z$ where y and z satisfy (10), then:

$$
\begin{aligned}
\lceil h_i(x)^\natural \rfloor &= \lceil h_i(y *_j z)^\natural \rfloor, & \\
&= \lceil (h_i(y) *_j h_i(z))^\natural \rfloor, & \text{(because } h \text{ is an } \omega\text{-functor)} \\
&= \lceil h_i(y)^\natural \rfloor + \lceil h_i(z)^\natural \rfloor, & \text{(by Lemma 36)} \\
&= \theta_i \lceil y \rfloor + \theta_i \lceil z \rfloor, & \text{(by the induction hypothesis)} \\
&= \theta_i (\lceil y \rfloor + \lceil z \rfloor), & \\
&= \theta_i \lceil y *_j z \rfloor, & \\
&= \theta_i \lceil x \rfloor. & \square
\end{aligned}
$$

Let us point out that $(T^*)^I$ is not free in general, so that h cannot be directly abelianized in the sense of Section 2.4.

We now turn to the proof of Proposition 3. Let $f, g : S^* \to T^*$ be ω-functors, such that $f \rightsquigarrow g$. There is a homotopy $h : S^* \to (T^*)^I$ from f to g, which, by Lemma 37 determines a family of maps

$$\theta_i : \mathbb{Z}S_i \to \mathbb{Z}T_{i+1}.$$

It turns out that $(\theta_i)_{i\geq 0}$ is a chain-homotopy between f^{ab} and g^{ab}. Indeed, if $x \in S^*_i$, we get an i-cylinder $h_i(x) : f_i(x) \curvearrowright g_i(x)$.

- If $i = 0$, we get $h_0(x)^\natural : f_0(x) \to g_0(x)$, so that $\partial_0\theta_0(\lceil x \rfloor) = \lceil g_0(x) \rfloor - \lceil f_0(x) \rfloor$, in other words

$$g^{\mathrm{ab}}_0 - f^{\mathrm{ab}}_0 = \partial_0 \circ \theta_0; \tag{11}$$

- if $i > 0$, we get $h_i(x) : h_{i-1}(\sigma_{i-1}x) \to h_{i-1}(\tau_{i-1}x) \mid f_i(x) \curvearrowright g_i(x)$. Lemma 35 applies, so that

$$
\begin{aligned}
&\lceil \sigma_i(h_i(x))^\natural \rfloor = \lceil f_i(x) \rfloor + \lceil (h_{i-1}(\tau_{i-1}x))^\natural \rfloor, \\
&\lceil \tau_i(h_i(x))^\natural \rfloor = \lceil (h_{i-1}(\sigma_{i-1}x))^\natural \rfloor + \lceil g_i(x) \rfloor.
\end{aligned}
$$

This implies

$$\lceil g_i(x) \rfloor - \lceil f_i(x) \rfloor = A + B$$

where

$$
\begin{aligned}
&A = \lceil \tau_i(h_i(x))^\natural \rfloor - \lceil \sigma_i(h_i(x))^\natural \rfloor = \partial_i \lceil (h_i(x))^\natural \rfloor = \partial_i\theta_i \lceil x \rfloor, \\
&B = \lceil (h_{i-1}(\tau_{i-1}x))^\natural \rfloor - \lceil (h_{i-1}(\sigma_{i-1}x))^\natural \rfloor.
\end{aligned}
$$

By Lemma 37, and the linearity of θ_{i-1},

$$
\begin{aligned}
B &= \theta_{i-1}(\lceil \tau_{i-1}x \rfloor) - \theta_{i-1}(\lceil \sigma_{i-1}x \rfloor), \\
&= \theta_{i-1}(\lceil \tau_{i-1}x \rfloor - \lceil \sigma_{i-1}x \rfloor), \\
&= \theta_{i-1}\partial_{i-1} \lceil x \rfloor.
\end{aligned}
$$

Hence

$$g^{\mathrm{ab}}_i - f^{\mathrm{ab}}_i = \partial_i \circ \theta_i + \theta_{i-1} \circ \partial_{i-1}. \tag{12}$$

Eqs. (11) and (12) exactly mean that θ is a chain-homotopy from f^{ab} to g^{ab}, thus proving the proposition:

$$
\begin{tikzcd}
\mathbb{Z}S_{i-1} \arrow[dr, "\theta_{i-1}"'] & \mathbb{Z}S_i \arrow[l, "\partial_{i-1}"'] \arrow[d, shift right, "f^{\mathrm{ab}}_i"'] \arrow[d, shift left, "g^{\mathrm{ab}}_i"] \arrow[dr, "\theta_i"] & \\
& \mathbb{Z}T_i & \mathbb{Z}T_{i+1} \arrow[l, "\partial_i"]
\end{tikzcd}
$$

Acknowledgement

We wish to thank the referee for many helpful suggestions, resulting in a much improved presentation of this work.

Appendix A. Counting generators

If $C : C_0 \leftleftarrows C_1 \leftleftarrows C_2 \cdots C_{n-1} \leftleftarrows C_n$ is an n-category and A is an (additive) abelian monoid, we consider the $n+1$-category $C^+ : C_0 \leftleftarrows C_1 \leftleftarrows C_2 \cdots C_{n-1} \leftleftarrows C_n \leftleftarrows C^+_{n+1}$ defined as follows:

- an $n+1$-cell in C^+ is a triple $(a, x, y) : x \to y$ where $a \in A$ and x, y are parallel n-cells in C;
- $(a, x, z) *_i (b, y, t) = (a + b, x *_i y, z *_i t)$ for $i < n$ and $x \triangleright_i y$ (so that $z \triangleright_i t$);

- $(a, x, y) *_n (b, y, z) = (a + b, x, z)$;
- $1_x = (0, x, x)$ for any n-cell x in C.

It is easy to see that those operations satisfy the laws of associativity, units, and interchange.
In particular, if $S_0^* \leftleftarrows S_1, S_1^* \leftleftarrows S_2, \ldots, S_{n-1}^* \leftleftarrows S_n, S_n^* \leftleftarrows S_{n+1}$ is an $n+1$-polygraph, we get:

- an n-category $C : S_0^* \leftleftarrows S_1^* \leftleftarrows S_2^* \cdots S_{n-1}^* \leftleftarrows S_n^*$ and an abelian monoid $A = \mathbb{Z}S_{n+1}$;
- an injection of S_{n+1} into C_{n+1}^+ mapping the generator $\xi : x \to y$ to the triple $(\lceil\xi\rfloor, x, y)$.

By the universal property, we get $\rho_n : S_{n+1}^* \to C_{n+1}^+$ which is compatible with sources, targets, products and units. This means that $\rho_n(u) = (\lceil u\rfloor, x, y)$ for any $n+1$-cell $u : x \to y$ in S^*, where $u \mapsto \lceil u\rfloor$ extends the canonical injection of S_{n+1} into $\mathbb{Z}S_{n+1}$ and satisfies the following properties:

$$\lceil u *_i v\rfloor = \lceil u\rfloor + \lceil v\rfloor \quad \text{for any } u \rhd_i v \text{ in } S_{n+1}^* \text{ with } i \le n, \qquad \lceil 1_x\rfloor = 0 \quad \text{for any } x \in S_n^*.$$

Appendix B. Decomposition

If M be a monoid and $D : M \leftleftarrows D_1$ is a category, we consider the monoid $\hat{D}$ defined as follows:

- an element of $\hat{D}$ is a pair $(\alpha, (u_\lambda)_{\lambda\in M})$ where $\alpha \in M$ and $(u_\lambda)_{\lambda\in M}$ is a family of cells $u_\lambda : \lambda \to \lambda\alpha$ in D;
- $(\alpha, (u_\lambda)_{\lambda\in M})(\beta, (v_\lambda)_{\lambda\in M}) = (\alpha\beta, (u_\lambda *_0 v_{\lambda\alpha})_{\lambda\in M})$.

It is easy to see that this operation is associative, with unit $(1, (1_\lambda)_{\lambda\in M})$.
In particular, if $f : S_1^* \to M$ is a morphism of monoid, we get:

- a category $D : M \leftleftarrows D_1$ where $D_1 = (M \cdot S_1)^*$;
- an injection of S_1 into $\hat{D}$ mapping the generator ξ to the pair $(\overline{\xi}, \langle\lambda \cdot \xi\rangle_{\lambda\in M})$.

By the universal property, we get a morphism $\rho : S_1^* \to \hat{D}$. This means that $\rho(x) = (\overline{x}, \langle\lambda \cdot x\rangle_{\lambda\in M})$ for all $x \in S_1^*$, where $\lambda \cdot x \mapsto \langle\lambda \cdot x\rangle$ extends the canonical inclusion of $M \cdot S_1$ into $(M \cdot S_1)^*$ and satisfies the following properties:

- we get $\langle\lambda \cdot x\rangle : \lambda \to \lambda\overline{x}$ for all $\lambda \in M$ and $x \in S_1^*$;
- $\langle\lambda \cdot xy\rangle = \langle\lambda \cdot x\rangle *_0 \langle\lambda\overline{x} \cdot y\rangle$ for all $\lambda \in M$ and $x, y \in S_1^*$;
- $\langle\lambda \cdot 1\rangle = 1_\lambda$ for all $\lambda \in M$.

Hence, we get the expected properties for S_1^*.
Now, let $C : \top \leftleftarrows C_1 \leftleftarrows C_2 \cdots C_{n-1} \leftleftarrows C_n$ be an n-monoid with $n > 0$ and assume that we have an $n+1$-category $D : M \leftleftarrows M \cdot C_1 \leftleftarrows M \cdot C_2 \cdots M \cdot C_{n-1} \leftleftarrows M \cdot C_n \leftleftarrows D_{n+1}$ extending the (partial) unfolding of $f : C \to M$. We consider the $n+1$-monoid $\hat{D} : \top \leftleftarrows C_1 \leftleftarrows C_2 \cdots C_{n-1} \leftleftarrows C_n \leftleftarrows \hat{D}_{n+1}$ defined as follows:

- an $n+1$-cell in $\hat{D}$ is a triple $((u_\lambda)_{\lambda\in M}, x, y) : x \to y$ where x, y are parallel n-cells in C and $(u_\lambda)_{\lambda\in M}$ is a family of $n+1$-cells $u_\lambda : \lambda \cdot x \to \lambda \cdot y$ in D;
- $((u_\lambda)_{\lambda\in M}, x, z)((v_\lambda)_{\lambda\in M}, y, t) = ((u_\lambda *_0 v_{\lambda\overline{x}})_{\lambda\in M}, xy, zt)$;
- $((u_\lambda)_{\lambda\in M}, x, z) *_i ((v_\lambda)_{\lambda\in M}, y, t) = ((u_\lambda *_i v_\lambda)_{\lambda\in M}, x *_i y, z *_i t)$ for $0 < i < n$ and $x \rhd_i y$ (so that $z \rhd_i t$);
- $((u_\lambda)_{\lambda\in M}, x, y) *_n ((v_\lambda)_{\lambda\in M}, y, z) = ((u_\lambda *_n v_\lambda)_{\lambda\in M}, x, z)$;
- $1_x = ((1_{\lambda\cdot x})_{\lambda\in M}, x, x)$ for any n-cell x in C.

It is easy to see that those operations satisfy the laws of associativity, units, and interchange.
In particular, if $\top \leftleftarrows S_1, S_1^* \leftleftarrows S_2, \ldots, S_{n-1}^* \leftleftarrows S_n, S_n^* \leftleftarrows S_{n+1}$ is an $n+1$-polygraph, we get:

- an n-monoid $C : \top \leftleftarrows S_1^* \leftleftarrows S_2^* \cdots S_{n-1}^* \leftleftarrows S_n^*$;
- an $n+1$-category $D : M \leftleftarrows M \cdot C_1 \leftleftarrows M \cdot C_2 \cdots M \cdot C_{n-1} \leftleftarrows M \cdot C_n \leftleftarrows D_{n+1}$ where $D_{n+1} = (M \cdot S_{n+1})^*$;
- an injection of S_{n+1} into $\hat{D}_{n+1}$ mapping the $n+1$-generator $\xi : x \to y$ to the triple $(\langle\lambda \cdot \xi\rangle_{\lambda\in M}, x, y)$.

By the universal property, we get $\rho_n : S_{n+1}^* \to \hat{D}_{n+1}$ which is compatible with sources, targets, products and units. This means that $\rho_n(u) = (\langle\lambda \cdot u\rangle_{\lambda\in M}, x, y)$ for any $n+1$-cell $u : x \to y$ in S^*, where $\lambda \cdot u \mapsto \langle\lambda \cdot u\rangle$ extends the canonical injection of $M \cdot S_{n+1}$ into $(M \cdot S_{n+1})^*$ and satisfies the following properties:

- we have $\langle\lambda \cdot u\rangle : \lambda \cdot x \to \lambda \cdot y$ for all $\lambda \in M$ and for any $n+1$-cell $u : x \to y$ in S^*;
- $\langle\lambda \cdot uv\rangle = \langle\lambda \cdot u\rangle *_0 \langle\lambda\overline{u} \cdot v\rangle$ for all $\lambda \in M$ and for any $n+1$-cells u, v in S^*;
- $\langle\lambda \cdot u *_i v\rangle = \langle\lambda \cdot u\rangle *_i \langle\lambda \cdot v\rangle$ for all $\lambda \in M$ and for any $n+1$-cells $u \rhd_i v$ in S^* with $0 < i \le n$;
- $\langle\lambda \cdot 1_x\rangle = 1_{\lambda\cdot x}$ for all $\lambda \in M$ and for any n-cell x in S^*.

Hence, we get the expected properties for S_{n+1}^* with $n > 0$.

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